Linear Elasticity Equations in Cylindrical Coordinates

Pavel Solin & Lenka Dubcova
hp-FEM group, University of Nevada, Reno
Academy of Sciences, Prague, Czech Republic
http://hpfem.math.unr.edu
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In this paper we derive the weak formulation of linear elasticity equations suitable for the finite element discretization of axisymmetric 3D problems.

Original equations in Cartesian coordinates

Let's start with some notations: By $\boldsymbol{u} = (u_1, u_2, u_3)^T$ we denote the displacement vector in 3D Cartesian coordinates, and by ϵ the tensor of small deformations,

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad 1 \le i, j \le 3.$$

The stress tensor σ has the form

$$\sigma_{ij} = \lambda \delta_{ij} \mathrm{div}\boldsymbol{u} + 2\mu\epsilon_{ij}, \quad 1 \le i, j \le 3, \tag{1}$$

where

$$\mathrm{div}\boldsymbol{u} = \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_k} = \sum_{k=1}^{3} \epsilon_{kk} = \mathrm{Tr}(\epsilon).$$

The symbols λ and μ are the Lamé constants and δ_{ij} is the Kronecker symbol ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise). The equilibrium equations have the form

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \quad 1 \le i \le 3, \tag{2}$$

where $(f_1, f_2, f_3)^T$ is the vector of internal forces (such as gravity).

The boundary conditions for linear elasticity are given by

$$\begin{aligned} u_i &= \hat{u}_i \quad \text{on } \Gamma_1 \\ \sum_{j=1}^{3} \sigma_{ij} n_j &= g_i \quad \text{on } \Gamma_2, \end{aligned}$$

where g_i are surface forces.

Weak formulation

Multiplying by test functions and integrating over the domain Ω we obtain

$$-\int_\Omega \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} v_i = \int_\Omega f_i \ v_i, \quad 1 \le i \le 3. \tag{3}$$

Using Green's theorem and the boundary conditions

$$\int_\Omega \sum_{j=1}^{3} \sigma_{ij} \frac{\partial v_i}{\partial x_j} - \int_{\partial\Omega} \sum_{j=1}^{3} \sigma_{ij} n_j v_i = \int_\Omega f_i \ v_i, \quad 1 \le i \le 3.$$

Thus

$$\int_\Omega \sum_{j=1}^{3} \sigma_{ij} \frac{\partial v_i}{\partial x_j} - \int_{\Gamma_2} g_i v_i = \int_\Omega f_i \ v_i, \quad 1 \le i \le 3. \tag{4}$$

Let us write the equations (4) in detail using relation (2)

$$\begin{aligned}
\int_\Omega \left[\lambda \mathrm{div} u + 2\mu \frac{\partial u_1}{\partial x_1}\right] \frac{\partial v_1}{\partial x_1} + \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \frac{\partial v_1}{\partial x_2} + \mu \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \frac{\partial v_1}{\partial x_3} - \int_{\Gamma_2} g_1 v_1 &= \int_\Omega f_1 \ v_1, \\
\int_\Omega \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \frac{\partial v_2}{\partial x_1} + \left[\lambda \mathrm{div} u + 2\mu \frac{\partial u_2}{\partial x_2}\right] \frac{\partial v_2}{\partial x_2} + \mu \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \frac{\partial v_2}{\partial x_3} - \int_{\Gamma_2} g_2 v_2 &= \int_\Omega f_2 \ v_2, \\
\int_\Omega \mu \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \frac{\partial v_3}{\partial x_1} + \mu \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \frac{\partial v_3}{\partial x_2} + \left[\lambda \mathrm{div} u + 2\mu \frac{\partial u_3}{\partial x_3}\right] \frac{\partial v_3}{\partial x_3} - \int_{\Gamma_2} g_3 v_3 &= \int_\Omega f_3 \ v_3.
\end{aligned} \tag{5}$$

Elementary transformation relations

First let us show how the partial derivatives of a scalar function g are transformed from Cartesian coordinates x_1, x_2, x_3 to cylindrical coordinates r, ϕ, z. Note that

$$x_1(r,\phi) = r\cos\phi, \quad x_2(r,\phi) = r\sin\phi, \quad x_3(z) = z.$$

Since

$$g(x_1, x_2, x_3) = g(x_1(r,\phi), x_2(r,\phi), x_3(z)),$$

it is

$$\begin{aligned}
\frac{\partial g}{\partial r} &= \frac{\partial g}{\partial x_1} \cos\phi + \frac{\partial g}{\partial x_2} \sin\phi, \\
\frac{\partial g}{\partial \phi} &= \frac{\partial g}{\partial x_1} (-r\sin\phi) + \frac{\partial g}{\partial x_2} r\cos\phi, \\
\frac{\partial g}{\partial z} &= \frac{\partial g}{\partial x_3}.
\end{aligned}$$

From here we obtain

$$\begin{aligned}\frac{\partial g}{\partial x_1} &= \frac{\partial g}{\partial r}\cos\phi - \frac{1}{r}\frac{\partial g}{\partial \phi}\sin\phi,\\ \frac{\partial g}{\partial x_2} &= \frac{\partial g}{\partial r}\sin\phi + \frac{1}{r}\frac{\partial g}{\partial \phi}\cos\phi,\\ \frac{\partial g}{\partial x_3} &= \frac{\partial g}{\partial z}.\end{aligned} \tag{6}$$

The relations between displacement components in Cartesian and cylindrical coordinates are

$$\begin{aligned}u_1 &= u_r\cos\phi,\\ u_2 &= u_r\sin\phi,\\ u_3 &= u_z.\end{aligned} \tag{7}$$

The same relations hold for surface forces g_i and volume forces f_i.
Applying (6) to u_1, we obtain

$$\begin{aligned}\frac{\partial u_1}{\partial x_1} &= \frac{\partial u_1}{\partial r}\cos\phi - \frac{1}{r}\frac{\partial u_1}{\partial \phi}\sin\phi,\\ \frac{\partial u_1}{\partial x_2} &= \frac{\partial u_1}{\partial r}\sin\phi + \frac{1}{r}\frac{\partial u_1}{\partial \phi}\cos\phi,\\ \frac{\partial u_1}{\partial x_3} &= \frac{\partial u_1}{\partial z}.\end{aligned}$$

Using (7) and the fact that u_r does not depend on ϕ, this yields

$$\begin{aligned}\frac{\partial u_1}{\partial x_1} &= \frac{\partial u_r}{\partial r}\cos^2\phi + \frac{1}{r}u_r\sin^2\phi,\\ \frac{\partial u_1}{\partial x_2} &= \frac{\partial u_r}{\partial r}\cos\phi\sin\phi - \frac{1}{r}u_r\cos\phi\sin\phi,\\ \frac{\partial u_1}{\partial x_3} &= \frac{\partial u_r}{\partial z}\cos\phi.\end{aligned}$$

Analogously, for u_2 we calculate

$$\begin{aligned}\frac{\partial u_2}{\partial x_1} &= \frac{\partial u_r}{\partial r}\cos\phi\sin\phi - \frac{1}{r}u_r\cos\phi\sin\phi,\\ \frac{\partial u_2}{\partial x_2} &= \frac{\partial u_r}{\partial r}\sin^2\phi + \frac{1}{r}u_r\cos^2\phi,\\ \frac{\partial u_2}{\partial x_3} &= \frac{\partial u_r}{\partial z}\sin\phi.\end{aligned}$$

For u_3, using that it does not depend on ϕ, we have

$$\begin{aligned}
\frac{\partial u_3}{\partial x_1} &= \frac{\partial u_z}{\partial r}\cos\phi, \\
\frac{\partial u_3}{\partial x_2} &= \frac{\partial u_z}{\partial r}\sin\phi, \\
\frac{\partial u_3}{\partial x_3} &= \frac{\partial u_z}{\partial z}.
\end{aligned}$$

For further reference, transform also $\mathrm{div} u$ into cylindrical coordinates

$$\begin{aligned}
\mathrm{div} u &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \\
&= \frac{\partial u_r}{\partial r}\cos^2\phi + \frac{1}{r}u_r\sin^2\phi + \frac{\partial u_r}{\partial r}\sin^2\phi + \frac{1}{r}u_r\cos^2\phi + \frac{\partial u_z}{\partial z} = \\
&= \frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{\partial u_z}{\partial z}
\end{aligned}$$

Axisymmetric formulation

Assuming that the domain Ω is axisymmetric, we can begin to transform the integrals in (5) to cylindrical coordinates. Recall that the Jacobian of the transformation is $J(r,\phi,z) = r$. The first equation in (5) has the form:

$$\begin{aligned}
&\int_\Omega r\left[\lambda(\frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{\partial u_z}{\partial z}) + 2\mu(\frac{\partial u_r}{\partial r}\cos^2\phi + \frac{1}{r}u_r\sin^2\phi)\right](\frac{\partial v_r}{\partial r}\cos^2\phi + \frac{1}{r}v_r\sin^2\phi) + \\
&r2\mu\left(\frac{\partial u_r}{\partial r}\cos\phi\sin\phi - \frac{1}{r}u_r\cos\phi\sin\phi\right)\left(\frac{\partial v_r}{\partial r}\cos\phi\sin\phi - \frac{1}{r}v_r\cos\phi\sin\phi\right) + \\
&r\mu\left(\frac{\partial u_r}{\partial z}\cos\phi + \frac{\partial u_z}{\partial r}\cos\phi\right)\frac{\partial v_r}{\partial z}\cos\phi - \int_{\Gamma_2} r g_r v_r \cos^2\phi = \int_\Omega r f_r\ v_r\cos^2\phi,
\end{aligned}$$

The second equation in (5) has the form:

$$\begin{aligned}
&\int_\Omega r2\mu\left(\frac{\partial u_r}{\partial r}\cos\phi\sin\phi - \frac{1}{r}u_r\cos\phi\sin\phi\right)\left(\frac{\partial v_r}{\partial r}\cos\phi\sin\phi - \frac{1}{r}v_r\cos\phi\sin\phi\right) + \\
&r\left[\lambda(\frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{\partial u_z}{\partial z}) + 2\mu(\frac{\partial u_r}{\partial r}\sin^2\phi + \frac{1}{r}u_r\cos^2\phi)\right](\frac{\partial v_r}{\partial r}\sin^2\phi + \frac{1}{r}v_r\cos^2\phi) + \\
&r\mu\left(\frac{\partial u_r}{\partial z}\sin\phi + \frac{\partial u_z}{\partial r}\sin\phi\right)(\frac{\partial v_r}{\partial z}\sin\phi) - \int_{\Gamma_2} r g_r v_r\sin^2\phi = \int_\Omega r f_r\ v_r\sin^2\phi,
\end{aligned}$$

Adding these two equations together we get

$$\int_\Omega r\lambda(\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r+\frac{\partial u_z}{\partial z})(\frac{\partial v_r}{\partial r}+\frac{1}{r}v_r)+$$
$$\int_\Omega r\mu\left[2\left(\frac{\partial u_r}{\partial r}\frac{\partial v_r}{\partial r}\cos^4\phi+\frac{1}{r}u_r\frac{\partial v_r}{\partial r}\sin^2\phi\cos^2\phi+\frac{1}{r}\frac{\partial u_r}{\partial r}v_r\sin^2\phi\cos^2\phi+\frac{1}{r^2}u_rv_r\sin^4\phi\right)+\right.$$
$$2\left(\frac{\partial u_r}{\partial r}\frac{\partial v_r}{\partial r}\sin^4\phi+\frac{1}{r}u_r\frac{\partial v_r}{\partial r}\sin^2\phi\cos^2\phi+\frac{1}{r}\frac{\partial u_r}{\partial r}v_r\sin^2\phi\cos^2\phi+\frac{1}{r^2}u_rv_r\cos^4\phi\right)+$$
$$4\left(\frac{\partial u_r}{\partial r}\frac{\partial v_r}{\partial r}\cos^2\phi\sin^2\phi-\frac{1}{r}u_r\frac{\partial v_r}{\partial r}\cos^2\phi\sin^2\phi-\frac{1}{r}\frac{\partial u_r}{\partial r}v_r\cos^2\phi\sin^2\phi+\frac{1}{r^2}u_rv_r\cos^2\phi\sin^2\phi\right)+$$
$$\left.\left(\frac{\partial u_r}{\partial z}\frac{\partial v_r}{\partial z}+\frac{\partial u_z}{\partial r}\frac{\partial v_r}{\partial z}\right)\right]-\int_{\Gamma_2}g_rv_rr=\int_\Omega f_r\ v_rr$$

This can be simplified to

$$\int_\Omega r\lambda(\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r+\frac{\partial u_z}{\partial z})(\frac{\partial v_r}{\partial r}+\frac{1}{r}v_r)+\int_\Omega r\mu\left[2\left(\frac{\partial u_r}{\partial r}\frac{\partial v_r}{\partial r}+\frac{1}{r^2}u_rv_r\right)+\left(\frac{\partial u_r}{\partial z}\frac{\partial v_r}{\partial z}+\frac{\partial u_z}{\partial r}\frac{\partial v_r}{\partial z}\right)\right]$$
$$-\int_{\Gamma_2}g_rv_rr=\int_\Omega f_r\ v_rr$$

Finally, the third equation in (8) has the form

$$\int_\Omega r\mu\left(\frac{\partial u_r}{\partial z}\cos\phi+\frac{\partial u_z}{\partial r}\cos\phi\right)\frac{\partial v_z}{\partial r}\cos\phi+r\mu\left(\frac{\partial u_r}{\partial z}\sin\phi+\frac{\partial u_z}{\partial r}\sin\phi\right)\frac{\partial v_z}{\partial r}\sin\phi+$$
$$r\left[\lambda(\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r+\frac{\partial u_z}{\partial z})+2\mu\frac{\partial u_z}{\partial z}\right]\frac{\partial v_z}{\partial z}-\int_{\Gamma_2}g_zv_zr=\int_\Omega f_z\ v_zr.$$

This gives us

$$\int_\Omega r\mu\left(\frac{\partial u_r}{\partial z}\frac{\partial v_z}{\partial r}+\frac{\partial u_z}{\partial r}\frac{\partial v_z}{\partial r}+2\frac{\partial u_z}{\partial z}\frac{\partial v_z}{\partial z}\right)+r\lambda\left(\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r+\frac{\partial u_z}{\partial z}\right)\frac{\partial v_z}{\partial z}-\int_{\Gamma_2}g_zv_zr=\int_\Omega f_z\ v_zr.$$

Since the integrands do not depend on ϕ, we can simplify this to integral over Ω_0, where Ω_0 is the intersection of the domain Ω with the $x_1^+x_3$ half-plane. Dividing both equations by 2π we get

$$\int_{\Omega_0} r\lambda(\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r+\frac{\partial u_z}{\partial z})(\frac{\partial v_r}{\partial r}+\frac{1}{r}v_r)+\int_{\Omega_0} r\mu\left[2\left(\frac{\partial u_r}{\partial r}\frac{\partial v_r}{\partial r}+\frac{1}{r^2}u_rv_r\right)+\left(\frac{\partial u_r}{\partial z}\frac{\partial v_r}{\partial z}+\frac{\partial u_z}{\partial r}\frac{\partial v_r}{\partial z}\right)\right]$$
$$-\int_{\Gamma_2}g_rv_rr=\int_{\Omega_0} f_r\ v_rr$$

$$\int_{\Omega_0} r\mu\left(\frac{\partial u_r}{\partial z}\frac{\partial v_z}{\partial r}+\frac{\partial u_z}{\partial r}\frac{\partial v_z}{\partial r}+2\frac{\partial u_z}{\partial z}\frac{\partial v_z}{\partial z}\right)+r\lambda\left(\frac{\partial u_r}{\partial r}+\frac{1}{r}u_r+\frac{\partial u_z}{\partial z}\right)\frac{\partial v_z}{\partial z}-\int_{\Gamma_2}g_zv_zr=\int_{\Omega_0} f_z\ v_zr.$$