Calculating Green Functions Using hp-FEM

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Abstract

Green functions are an excellent tool for working with a solution to any ODE or PDE. In this text we explain how it works and then show how one can calculate them using hp-FEM.

1 Introduction

Let's put any ODE or PDE in the form:

$$Lu(x) = f(x) \tag{1}$$

Here L is a differential operator and x can have any dimension, e.g. 1D (ODE), 2D, 3D or more (PDE). Then we can express the solution as

$$u(x) = L^{-1}f(x) = \int G(x, x')f(x')\mathrm{d}x' \tag{2}$$

where $G(x, x')$ is a Green function, that needs to satisfy the equation:

$$LG(x, x') = \delta(x - x') \tag{3}$$

Remember, that L acts on x only, so we can check, that (2) indeed solves the PDE (1):

$$Lu(x) = L\int G(x, x')f(x')\mathrm{d}x' = \int LG(x, x')f(x')\mathrm{d}x' = \int \delta(x - x')f(x')\mathrm{d}x' = f(x)$$

2 Boundary Conditions

The equation (3) doesn't determine the Green function uniquely, because one can add to it any solution of the homogeneous equation $Lu(x) = 0$. We can use this freedom to solve (3) for any boundary condition, for example using hp-FEM (more on that below). So we prescribe a boundary condition and find the Green function (by solving (3)) that satisfies the boundary condition. It can be shown, that $u(x)$ determined from (2) then also needs to satisfy the same boundary condition.

3 Examples

3.1 Laplace Equation

$$\nabla^2 u(x) = f(x)$$
$$\nabla^2 G(x,x') = \delta(x-x')$$

with boundary condition $G(x) = 0$ at infinity. Then:

$$G(x,x') = -\frac{1}{4\pi}\frac{1}{|x-x'|}$$

and

$$u(x) = -\frac{1}{4\pi}\int \frac{f(x')}{|x-x'|}\mathrm{d}x'$$

3.2 Helmholtz Equation in 3D

$$(\nabla^2 + k^2)u(x) = f(x)$$
$$(\nabla^2 + k^2)G(x,x') = \delta(x-x')$$

with boundary condition $G(x) = 0$ at infinity. Then:

$$G(x,x') = -\frac{1}{4\pi}\frac{e^{ik|x-x'|}}{|x-x'|}$$

and

$$u(x) = -\frac{1}{4\pi}\int \frac{f(x')e^{ik|x-x'|}}{|x-x'|}\mathrm{d}x'$$

3.3 Helmholtz Equation in 1D

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1\right)u(x) = f(x)$$
$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 1\right)G(x,x') = \delta(x-x')$$

with boundary conditions $u(0) = u(\frac{\pi}{2}) = 0$. Then:

$$G(x,x') = \begin{cases} -\sin x \cos x' & x < x' \\ -\cos x \sin x' & x > x' \end{cases}$$

and

$$u(x) = \int G(x,x')f(x')\mathrm{d}x' = -\cos x \int_0^x f(x')\sin x'\mathrm{d}x' - \sin x \int_x^{\frac{\pi}{2}} f(x')\cos x'\mathrm{d}x'$$

To show that this really works, let's take for example $f(x) = 3\sin 2x$. Then

$$u(x) = -\cos x \int_0^x 3\sin 2x' \sin x' \mathrm{d}x' - \sin x \int_x^{\frac{\pi}{2}} 3\sin 2x' \cos x' \mathrm{d}x'$$

We can use for example SymPy to evaluate the integrals:

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In [1]: u = -cos(x)*integrate(3*sin(2*y)*sin(y), (y, 0, x)) - \
    sin(x)*integrate(3*sin(2*y)*cos(y), (y, x, pi/2))

In [2]: u
Out[2]:
-(cos(x)*sin(2*x) - 2*cos(2*x)*sin(x))*cos(x) - (sin(x)*sin(2*x)
    + 2*cos(x)*cos(2*x))*sin(x)

In [3]: simplify(u)
Out[3]:
     2                  2
- cos (x)*sin(2*x) - sin (x)*sin(2*x)

In [4]: trigsimp(_)
Out[4]: -sin(2*x)
```

And we get

$$u(x) = -\sin 2x$$

We can easily check, that $u'' + u = 3\sin 2x = f(x)$, so all is ok.

4 Finite Element Method

Let's show it on the Laplace equation. We want to solve:

$$\nabla^2 G(x, x') = \delta(x - x')$$

We will treat x' as a parameter, so we define $g_{x'}(x) \equiv G(x, x')$:

$$\nabla^2 g_{x'}(x) = \delta(x - x')$$

We set $g_{x'}(x) = 0$ on the boundary and we get:

$$-\int \nabla g_{x'}(x) \cdot \nabla v(x) \mathrm{d}x = \int v(x)\delta(x - x')\mathrm{d}x$$

$$-\int \nabla g_{x'}(x) \cdot \nabla v(x) \mathrm{d}x = v(x')$$

So we choose x' and then solve for $g_{x'}(x)$ using hp-FEM and we get the Green function $G(x, x')$ for all x and one particular x'. We can then evaluate the integral (2) numerically – one would have to use FEM for all x' that are needed in the integral, so that is not efficient, but it should work. One will then be able to play with Green functions and be able to calculate them numerically for any boundary condition (which is not possible analytically).