Equation Systems with Linear Constraints

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Problem: solve

$$Ax = b$$

subject to the linear constraint

$$c^T x = d.$$

A is an $n \times n$ SPD matrix, b, c, x are n-vectors and d is a scalar.

Solution: Lagrange multiplier method. Define the Lagrangian

$$\Lambda(x, \lambda) = f(x) + \lambda g(x).$$

Here, $f(x)$ is a function to be minimized/maximized and $g(x) = 0$ is a constraint which has to be satisfied. Both is attained if and only if

$$\nabla \Lambda(x, \lambda) = 0 \tag{1}$$

In our case, we will be minimizing the function

$$f(x) = \frac{1}{2} x^T A x - x^T b,$$

subject to the constraint

$$g(x) = c^T x - d = 0.$$

Provided that A is SPD, it can be shown that $f(x)$ is always nonnegative and moreover it attains zero if and only if $Ax = b$, thus it makes sense to minimize it.

We now have

$$\nabla_x \Lambda(x, \lambda) = Ax - b + \lambda c$$

and

$$\nabla_\lambda \Lambda(x, \lambda) = c^T x - d.$$

Equation (1) then has the form

$$\begin{bmatrix} A & c \\ c^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}. \tag{2}$$

Solving this linear system solves the original problem. The system will be consistent as long as the set $f(x) = 0$ coincides with $c^T x - d = 0$ at only one point (TODO: prove this). It is straightforward to extend this to an arbitrary number of constraints ($Cx = d$).

The downside of this method is that the matrix in (2) is indefinite because of the zero(s). This is not a problem for direct solvers, though.