Interface Tracking via Level Sets on Dynamical *hp*-Meshes

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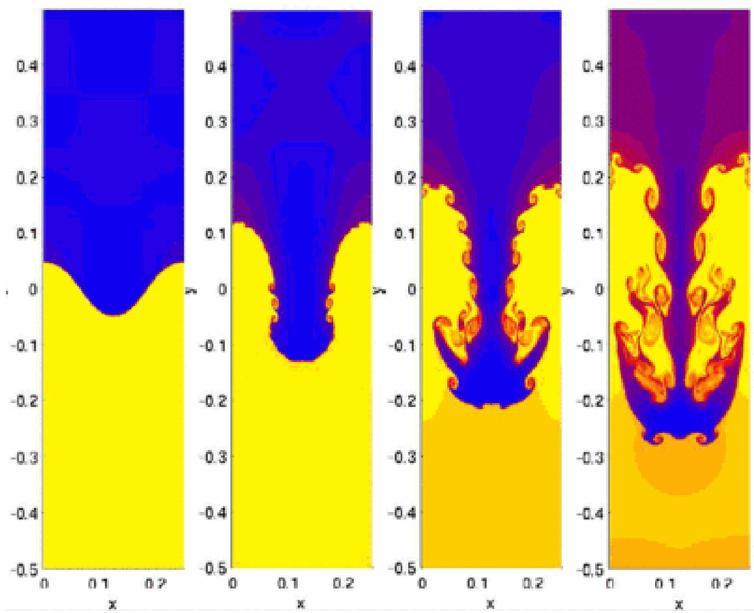

Figure 1: Example of the Raleigh-Taylor instability.

The above example shows the so-called *Raleigh-Taylor instability* between two incompressible viscous liquids. Initially, the heavier liquid is on top of the lighter one. Because of gravity, both liquids start to move and the motion continues until a steady state is reached (the lighter liquid is on top of the heavier one). This is a nice model example for interface-tracking algorithms.

For simplicity, let the domain Ω where the process takes place be a unit square. All four edges of the square represent solid walls with zero velocity boundary condition $\boldsymbol{v}(\boldsymbol{x}, t) = 0$ on $\partial\Omega$. Initially, Ω is divided into two subdomains Ω_1 and Ω_2 representing the initial configuration of the liquids, as shown in Fig. 2. The initial velocity $\boldsymbol{v}(\boldsymbol{x}, 0) = 0$ for all $\boldsymbol{x} \in \Omega$, and the initial density $\rho(\boldsymbol{x}, 0)$ is piecewise constant: ρ_1 in Ω_1 and ρ_2 in Ω_2, with ρ_1, ρ_2.

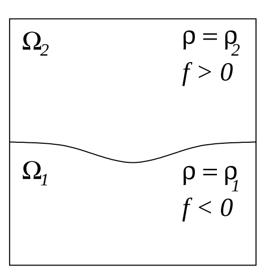

Figure 2: Initial configuration.

Level Set Function

We will use the so-called *level set function* $f(\boldsymbol{x}, t)$ to distinguish between the two liquids. This is a real-valued function defined in $\Omega \times [0, T]$. Initially, f is chosen in such a way that $f(\boldsymbol{x}, 0) < 0$ for all $\boldsymbol{x} \in \Omega_1$ and $f(\boldsymbol{x}, 0) > 0$ for all $\boldsymbol{x} \in \Omega_2$. Thus the interface between the two liquids is defined implicitly as the zero level set of f. The reader can imagine f, for example, as a color of the liquids. When the liquids start to move, all fluid particles carry their color with them. Thus the level set function will evolve with the flow and the zero level set of $f(\boldsymbol{x}, t)$ will represent the interface at all times.

The function $f(\boldsymbol{x}, t)$ obeys the differential equation

$$\frac{\partial f}{\partial t} + \mathrm{div}(f\boldsymbol{v}) = 0 \tag{1}$$

which is obtained easily using the Reynolds transport theorem. Using the incompressibility condition $\mathrm{div}\boldsymbol{v} = 0$, equation (1) can be simplied to

$$\frac{\partial f}{\partial t} + \nabla f \cdot \boldsymbol{v} = 0, \tag{2}$$

but this may not be as good as (1) for finite element approximation.

Navier-Stokes Equations for Two Liquids

We use the standard form of the Navier-Stokes equations,

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \right) = -\nabla p + \nu \Delta \boldsymbol{v} + \boldsymbol{g}, \tag{3}$$

where ν is the viscosity and $\boldsymbol{g} = (0, -10\rho)$ the density of gravitational forces. Of course, this is supplemented with the incompressibility condition $\mathrm{div}\boldsymbol{v} = 0$. As opposed to the standard Navier-Stokes equations, however, the density is a function of space and time:

$$\rho(\boldsymbol{x}, t) = \begin{cases} \rho_1 \text{ if } f(\boldsymbol{x}, t) < 0, \\ \\ \rho_2 \text{ if } f(\boldsymbol{x}, t) > 0. \end{cases} \tag{4}$$

Note that also the density of gravitational forces $\boldsymbol{g}$ depends on $f(\boldsymbol{x}, t)$.

Coupling

Thus the system (1), (3) is coupled both ways: The level set function determines the density and thus influences the flow, and the flow changes the level set function.

Suggestion how to start

Define a fine Cartesian grid in Ω, for example of 100×100 square elements of first order. Define the initial values of the function f in some way that corresponds to Fig. 2 (for first-order elements, defining f via vertex values is sufficient). Run the computation without automatic adaptivity first.