

Newton's Method for Incompressible Navier-Stokes Equations

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Abstract

This paper explains how the Newton's method can be applied to time-dependent incompressible Navier-Stokes equations. We assume that the reader knows how the Newton's method works, so the main part of this document describes the bilinear forms that are necessary to evaluate the Jacobian matrix DF/DY of the nonlinear discrete problem $F(Y) = 0$. For simplicity, we will use the implicit Euler method in time. The forms presented in this paper can be extended to the Crank-Nicolson and other time stepping methods easily.

1 Governing Equations and Weak Formulation

Discretized by means of the implicit Euler method in time, the Navier-Stokes equations have the form

$$\begin{aligned} \frac{u^{n+1} - u^n}{\tau} - \frac{1}{Re} \Delta u^{n+1} + u^{n+1} \frac{\partial u^{n+1}}{\partial x} + v^{n+1} \frac{\partial u^{n+1}}{\partial y} + \frac{\partial p}{\partial x} &= 0, \\ \frac{v^{n+1} - v^n}{\tau} - \frac{1}{Re} \Delta v^{n+1} + u^{n+1} \frac{\partial v^{n+1}}{\partial x} + v^{n+1} \frac{\partial v^{n+1}}{\partial y} + \frac{\partial p}{\partial y} &= 0, \\ \frac{\partial u^{n+1}}{\partial x} + \frac{\partial v^{n+1}}{\partial y} &= 0. \end{aligned}$$

Boundary conditions are chosen as follows: None for p since this is an L^2 -function. For the velocity components u, v we consider inlet (u, v given), no-slip solid wall ($u = v = 0$), or do-nothing boundary condition at the outlet (surface integrals coming out of the Green's theorem are neglected). In this way, there are no surface integrals in the weak formulation.

The finite element expansion of the solution components u^{n+1}, v^{n+1} and p^{n+1} is

$$u^{n+1} = \sum_{k=1}^{N^u} y_k^u \varphi_k^u, \quad v^{n+1} = \sum_{k=1}^{N^v} y_k^v \varphi_k^v, \quad p^{n+1} = \sum_{k=1}^{N^p} y_k^p \varphi_k^p,$$

where $\varphi_i^u, \varphi_i^v, \varphi_i^p$ are the basis functions for u, v and p . Hence the global coefficient vector Y consists of three parts Y^u, Y^v, Y^p corresponding to the unknowns u^{n+1}, v^{n+1} and p^{n+1} , respectively, and its length is $N = N^u + N^v + N^p$.

The finite element weak formulation reads

$$\begin{aligned}
F_i^u(Y^{n+1}) &= \int_{\Omega} \frac{u^{n+1} - u^n}{\tau} \varphi_i^u + \int_{\Omega} \frac{1}{Re} \nabla u^{n+1} \cdot \nabla \varphi_i^u \\
&\quad + \int_{\Omega} u^{n+1} \frac{\partial u^{n+1}}{\partial x} \varphi_i^u + \int_{\Omega} v^{n+1} \frac{\partial u^{n+1}}{\partial y} \varphi_i^u - \int_{\Omega} p^{n+1} \frac{\partial \varphi_i^u}{\partial x} = 0, \quad i = 1, 2, \dots, N^u, \\
F_i^v(Y^{n+1}) &= \int_{\Omega} \frac{v^{n+1} - v^n}{\tau} \varphi_i^v + \int_{\Omega} \frac{1}{Re} \nabla v^{n+1} \cdot \nabla \varphi_i^v \\
&\quad + \int_{\Omega} u^{n+1} \frac{\partial v^{n+1}}{\partial x} \varphi_i^v + \int_{\Omega} v^{n+1} \frac{\partial v^{n+1}}{\partial y} \varphi_i^v - \int_{\Omega} p^{n+1} \frac{\partial \varphi_i^v}{\partial y} = 0, \quad i = 1, 2, \dots, N^v, \\
F_i^p(Y^{n+1}) &= \int_{\Omega} \frac{\partial u^{n+1}}{\partial x} \varphi_i^p + \int_{\Omega} \frac{\partial v^{n+1}}{\partial y} \varphi_i^p = 0, \quad i = 1, 2, \dots, N^p.
\end{aligned}$$

Thus we have a system of N nonlinear algebraic equations of the form $F(Y) = 0$, where the vector-valued function F consists of three parts F^u, F^v and F^p corresponding to u^{n+1}, v^{n+1} and p^{n+1} , respectively.

2 Newton's Method

The Newton's method can be written as follows,

$$J(Y^n) \delta Y^{n+1} = -F(Y^n),$$

where $J = DF/DY$, Y^0 is a given initial guess, and $Y^{n+1} = Y^n + \delta Y^{n+1}$. The Jacobian matrix J has a 3×3 block structure

$$J(Y) = \begin{pmatrix} \frac{\partial F_i^u}{\partial y_j^u} & \frac{\partial F_i^u}{\partial y_j^v} & \frac{\partial F_i^u}{\partial y_j^p} \\ \frac{\partial F_i^v}{\partial y_j^u} & \frac{\partial F_i^v}{\partial y_j^v} & \frac{\partial F_i^v}{\partial y_j^p} \\ \frac{\partial F_i^p}{\partial y_j^u} & \frac{\partial F_i^p}{\partial y_j^v} & \frac{\partial F_i^p}{\partial y_j^p} \end{pmatrix}.$$

Before we calculate the blocks, let us realize that

$$u^{n+1} = u(Y^{n+1}) = u(Y^{u,n+1}) = \sum_{k=1}^{N^u} y_k^u \varphi_k^u$$

and thus

$$\frac{\partial u^{n+1}}{\partial y_j^u} = \varphi_j^u, \quad \frac{\partial \nabla u^{n+1}}{\partial y_j^u} = \nabla \varphi_j^u, \quad \frac{\partial \nabla u^{n+1}}{\partial y_j^v} = \frac{\partial \nabla u^{n+1}}{\partial y_j^p} = 0,$$

etc. The blocks of $J(Y^n)$ have the form

$$\begin{aligned}\frac{\partial F_i^u}{\partial y_j^u}(Y^n) &= \int_{\Omega} \frac{1}{\tau} \varphi_j^u \varphi_i^u + \frac{1}{Re} \int_{\Omega} \nabla \varphi_j^u \cdot \nabla \varphi_i^u + \int_{\Omega} \left[\varphi_j^u \frac{\partial u^n}{\partial x} + u^n \frac{\partial \varphi_j^u}{\partial x} \right] \varphi_i^u + \int_{\Omega} v^n \frac{\partial \varphi_j^u}{\partial y} \varphi_i^u, \\ \frac{\partial F_i^u}{\partial y_j^v}(Y^n) &= \int_{\Omega} \varphi_j^v \frac{\partial u^n}{\partial y} \varphi_i^u, \\ \frac{\partial F_i^u}{\partial y_j^p}(Y^n) &= - \int_{\Omega} \varphi_j^p \frac{\partial \varphi_i^u}{\partial x}.\end{aligned}$$

The second block-row contains the blocks

$$\begin{aligned}\frac{\partial F_i^v}{\partial y_j^v}(Y^n) &= \int_{\Omega} \varphi_j^u \frac{\partial v^n}{\partial x} \varphi_i^v, \\ \frac{\partial F_i^v}{\partial y_j^u}(Y^n) &= \int_{\Omega} \frac{1}{\tau} \varphi_j^v \varphi_i^v + \frac{1}{Re} \int_{\Omega} \nabla \varphi_j^v \cdot \nabla \varphi_i^v + \int_{\Omega} u^n \frac{\partial \varphi_j^v}{\partial x} \varphi_i^v + \int_{\Omega} \left[\varphi_j^v \frac{\partial v^n}{\partial y} + v^n \frac{\partial \varphi_j^v}{\partial y} \right] \varphi_i^v, \\ \frac{\partial F_i^v}{\partial y_j^p}(Y^n) &= - \int_{\Omega} \varphi_j^p \frac{\partial \varphi_i^v}{\partial y}.\end{aligned}$$

Finally, the last block-row has the form

$$\begin{aligned}\frac{\partial F_i^p}{\partial y_j^u}(Y^n) &= \int_{\Omega} \frac{\partial \varphi_j^u}{\partial x} \varphi_i^p, \\ \frac{\partial F_i^p}{\partial y_j^v}(Y^n) &= \int_{\Omega} \frac{\partial \varphi_j^v}{\partial y} \varphi_i^p, \\ \frac{\partial F_i^p}{\partial y_j^p}(Y^n) &= 0.\end{aligned}$$