

FEM for ODE and DDE

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Abstract

We describe how the finite element method (FEM) can be used to solve initial-value problems for first-order ordinary differential equations (ODE). At the end we discuss the extension to delay differential equations (DDE).

1 Weak formulation

Consider a first-order ODE in the form

$$x'(t) = f(x(t), t) \quad \text{in } (0, T) \quad (1)$$

equipped with an initial condition

$$x(0) = x_0. \quad (2)$$

Since $x_0 \neq 0$ in general, we define a Dirichlet lift function $D(t)$ as any bounded continuous function such that $D(0) = x_0$. Typically, $D(t)$ is chosen in such a way that it drops fast (and linearly) from x_0 to zero and remains zero in the rest of the interval. Instead of looking for a solution $x(t)$ satisfying (2), we decompose $x(t)$ to

$$x(t) = D(t) + \tilde{x}(t)$$

and look for a new solution $\tilde{x}(t)$ satisfying the zero Dirichlet condition

$$\tilde{x}(0) = 0. \quad (3)$$

The advantage of introducing $\tilde{x}(t)$ is that all continuous functions in $(0, T)$ satisfying (3) form a linear space, as opposed to functions satisfying (2). This is a standard way nonzero Dirichlet boundary conditions are handled in FEM.

By multiplying (1) with a test function v satisfying (3) and integrating over the interval $(0, T)$, we arrive at the weak formulation for $\tilde{x}(t)$: Find $\tilde{x} \in V$ such that

$$\int_0^T \tilde{x}'(t)v(t) + D'(t)v(t) - f(x(t), t)v(t) dt = 0 \quad \text{for all } v \in V, \quad (4)$$

where

$$V = \{v \in H^1(0, T), v(0) = 0\}.$$

2 Discretization

In the standard way, we cover the time interval $(0, T)$ with a mesh $\tau_{h,p}$ consisting of M nonoverlapping subintervals K_1, K_2, \dots, K_M . Every subinterval K_i is equipped with a polynomial order $p_i \geq 1$. We define the standard subspace $V_{h,p} \subset V$ of continuous, piecewise-polynomial functions over the mesh $\tau_{h,p}$. This subspace has finite dimension

$$N = \dim(V_{h,p}) = \sum_{m=1}^M p_m.$$

Next we choose a suitable basis of the space $V_{h,p}$ consisting of functions v_1, v_2, \dots, v_N . For example, if all elements are linear ($p_i = 1$) then the basis will consist of M "hat" functions only.

We express the approximate solution as

$$\tilde{x}_{h,p}(t) = \sum_{j=1}^N y_j v_j(t) \quad (5)$$

where y_1, y_2, \dots, y_N are unknown coefficients. The Dirichlet lift $D(t)$ is chosen to drop linearly from x_0 to zero within the first element K_1 , and to be zero in all remaining elements. Again

$$x_{h,p}(t) = D(t) + \tilde{x}_{h,p}(t).$$

The weak formulation (4) is restricted to the subspace $V_{h,p}$: Find $\tilde{x}_{h,p} \in V_{h,p}$ such that

$$\int_0^T \tilde{x}'_{h,p}(t)v(t) + D'(t)v(t) - f(x_{h,p}(t), t)v(t) dt = 0 \quad \text{for all } v \in V_{h,p}.$$

This is equivalent to

$$\int_0^T \tilde{x}'_{h,p}(t)v_i(t) + D'(t)v_i(t) - f(x_{h,p}(t), t)v_i(t) dt = 0 \quad \text{for } i = 1, 2, \dots, N. \quad (6)$$

Plugging (5) into (6), we have

$$\int_0^T \sum_{j=1}^N y_j v'_j(t)v_i(t) + D'(t)v_i(t) - f\left(D(t) + \sum_{j=1}^N y_j v_j(t), t\right) v_i(t) dt = 0 \quad \text{for } i = 1, 2, \dots, N. \quad (7)$$

Hence we obtained a system of nonlinear algebraic equations

$$\Phi(Y) = \mathbf{0} \quad (8)$$

for the unknown coefficient vector Y . Here, $\mathbf{0}$ stands for the zero vector and the i th component of the nonlinear vector-valued function $\Phi(Y)$ has the form

$$\Phi_i(Y) = \int_0^T \sum_{j=1}^N y_j v'_j(t)v_i(t) + D'(t)v_i(t) - f\left(D(t) + \sum_{j=1}^N y_j v_j(t), t\right) v_i(t) dt.$$

System (8) will be solved using the Newton's method. For that we need the Jacobi matrix of Φ :

$$\left(\frac{D\Phi}{DY}\right)_{ik} = \frac{\partial\Phi_i}{\partial y_k} = \int_0^T v'_k(t)v_i(t) - \frac{\partial f}{\partial x} \left(D(t) + \sum_{j=1}^N y_j v_j(t), t \right) v_k(t)v_i(t) dt. \quad (9)$$

The initial condition for the Newton's iteration can be chosen, for example, as $Y^{(0)} = 0$ (which corresponds to $x_{h,p}^{(0)} = D$).

Remark: We need to try the method first to see how it actually works. But we expect that it should work and moreover that employment of automatic adaptivity to this problem might bring interesting new problems. Namely, the error here is nonlocal, propagating from the left to the right. Usual adaptive methods are not able to handle nonlocal error sources. So, this could be a nice new material to study adaptive methods!

3 Extension to DDE

Next let's discuss the extension of the FEM approach described above to DDE of the form

$$x'(t) = f(x(t - \tau), t) \quad t \in (0, T) \quad (10)$$

equipped with the initial condition

$$x(t) = x_0(t) \quad t \in (-\tau, 0). \quad (11)$$

Here, x_0 is a function defined in the interval $(-\tau, 0]$ (as opposed to (2) where it was just a number). Nevertheless, the unknown function $x(t)$ also is defined in the interval $(0, T)$ and it needs to satisfy an initial condition analogous to (2),

$$x(0) = x_0(0). \quad (12)$$

The Dirichlet lift $D(t)$ is defined as $x_0(t)$ for $t \in (-\tau, 0]$ and extended into $(0, T)$ as in the previous case. The solution $x(t)$ is decomposed into

$$x(t) = D(t) + \tilde{x}(t)$$

as in the previous case. The weak formulation for $\tilde{x}(t)$ reads: Find $\tilde{x} \in V$ such that

$$\int_0^T \tilde{x}'(t)v(t) + D'(t)v(t) - f(x(t - \tau), t)v(t) dt = 0 \quad \text{for all } v \in V, \quad (13)$$

where

$$V = \{v \in H^1(0, T), v(0) = 0\}.$$

This can be discretized in the same way as (4),

$$\int_0^T \sum_{j=1}^N y_j v_j'(t) v_i(t) + D'(t) v_i(t) - f \left(D(t - \tau) + \sum_{j=1}^N y_j v_j(t - \tau), t \right) v_i(t) dt = 0 \quad \text{for } i = 1, 2, \dots, N. \quad (14)$$

Hence we have again a system of nonlinear algebraic equations

$$\Phi(Y) = \mathbf{0} \quad (15)$$

where

$$\Phi_i(Y) = \int_0^T \sum_{j=1}^N y_j v_j'(t) v_i(t) + D'(t) v_i(t) - f \left(D(t - \tau) + \sum_{j=1}^N y_j v_j(t - \tau), t \right) v_i(t) dt.$$

The Jacobi matrix of Φ has entries

$$\left(\frac{D\Phi}{DY} \right)_{ik} = \frac{\partial \Phi_i}{\partial y_k} = \int_0^T v_k'(t) v_i(t) - \frac{\partial f}{\partial x} \left(D(t - \tau) + \sum_{j=1}^N y_j v_j(t - \tau), t \right) v_k(t - \tau) v_i(t) dt. \quad (16)$$

The initial condition for the Newton's iteration can be chosen again as $Y^{(0)} = 0$.

Remark: Although the ODE and DDE case appear almost the same on paper, there will be some implementation differences. The delay $t - \tau$ will change the way the test and basis functions interact. For example, a pair v_k, v_i such that

$$\text{supp} v_k(t) \cap \text{supp} v_i(t) \neq \emptyset$$

which generated a nonzero entry ki in the Jacobi matrix (9) will not contribute anything to (16) if

$$\text{supp} v_k(t - \tau) \cap \text{supp} v_i(t) = \emptyset,$$

and vice versa.