New Volumetric Weak Forms for Fast Product Quadrature

Pavel Solin
hp-FEM group, University of Nevada, Reno
Academy of Sciences, Prague, Czech Republic
http://hpfem.org/, November 2009

Abstract

I suggest to enable one more way of defining volumetric weak forms that would speed up assembling on undistorted quads/hexes.

1 Main Idea

Currently, we use full 2D/3D quadrature for all integrals in volumetric weak forms and for every mesh element K. In many cases, however, the element has a product geometry (quad or hex with all right angles, can be rotated), the shape functions have product form $u(x_1, x_2) = u_1(x_1)u_2(x_2)$ (2D) or $u(x_1, x_2, x_3) = u_1(x_1)u_2(x_2)u_3(x_3)$ (3D), and the form itself is a product such as, for example,

$$\int_K uv \, \mathrm{d}x_1 \mathrm{d}x_2, \quad \int_K \frac{\partial u}{\partial x_1} v \, \mathrm{d}x_1 \mathrm{d}x_2, \quad \int_K \nabla u \cdot \nabla v \, \mathrm{d}x_1 \mathrm{d}x_2.$$

If all these conditions are met, we could save a lot on quadrature by decomposing the 2D/3D integrals into 1D integrals in spatial directions.

Let me describe this on an example of the weak form for the Laplace operator in 2D (last one above). Currently, this form is written as

```
double val = 0;
for(int i = 0; i<num; i++) {
  val += (dudx[i]*dvdx[i] + dudy[i]*dvdy[i]) * weights[i];
}
return val;
```

where `num` is the number of 2D integration points in the reference square/triangle.

However, if the basis function u and the test function v have product forms

$$u(x_1, x_2) = u_1(x_1)u_2(x_2), \quad v(x_1, x_2) = v_1(x_1)v_2(x_2)$$

and if the quad is undistorted (Jacobian J_K of reference map is constant), this leads on the reference square $\hat{K} = (-1, 1)^2$ to the integral

$$\int_K \nabla u \cdot \nabla v \, \mathrm{d}x_1 \mathrm{d}x_2 = \int_{\hat{K}} |J_K| \, \nabla \tilde{u} \cdot \nabla \tilde{v} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2$$

$$= |J_K| \int_{-1}^{1} \int_{-1}^{1} \frac{\partial \tilde{u}_1}{\partial x_1}(\xi_1) \tilde{u}_2(\xi_2) \frac{\partial \tilde{v}_1}{\partial \xi_1}(\xi_1) \tilde{v}_2(\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2$$

$$+ |J_K| \int_{-1}^{1} \int_{-1}^{1} \tilde{u}_1(\xi_1) \frac{\partial \tilde{u}_2}{\partial \xi_2}(\xi_2) \tilde{v}_1(\xi_1) \frac{\partial \tilde{v}_2}{\partial \xi_2}(\xi_2) \mathrm{d}\xi_1 \mathrm{d}\xi_2$$

$$= |J_K| \int_{-1}^{1} \frac{\partial \tilde{u}_1}{\partial \xi_1}(\xi_1) \frac{\partial \tilde{v}_1}{\partial \xi_1}(\xi_1) \mathrm{d}\xi_1 \int_{-1}^{1} \tilde{u}_2(\xi_2) \tilde{v}_2(\xi_2) \mathrm{d}\xi_2$$

$$+ |J_K| \int_{-1}^{1} \tilde{u}_1(\xi_2) \tilde{v}_1(\xi_2) \mathrm{d}\xi_2 \int_{-1}^{1} \frac{\partial \tilde{u}_2}{\partial \xi_1}(\xi_1) \frac{\partial \tilde{v}_2}{\partial \xi_1}(\xi_1) \mathrm{d}\xi_1.$$

The corresponding code would look as follows:

```
double val1 = 0, val2 = 0, val3 = 0, val4 = 0;
for(int i = 0; i<num_1D_max; i++) {
  val1 += u1[i] * v1[i] * weights[i];
  val2 += u2[i] * v2[i] * weights[i];
  val3 += du1dx[i] * dv1dx[i] * weights[i];
  val4 += du2dx[i] * dv2dx[i] * weights[i];
}
return val3*val2 + val1*val4
```

The arguments of the new volumetric product form would include:

`num_1D_max`... number of int. pts corresp. to the max. dir. pol. deg. on the element K,
`x[]`... 1D integration points,
`w[]`... corresponding weights,
`u1[]`... values of the 1D shape function used for u_1,
`u2[]`... values of the 1D shape function used for u_2,
`v1[]`... values of the 1D shape function used for v_1,
`v2[]`... values of the 1D shape function used for v_2,
`du1dx[]`... values of the 1D shape function used for $\partial u_1/\partial x_1$,
`du2dx[]`... values of the 1D shape function used for $\partial u_2/\partial x_2$,
`dv1dx[]`... values of the 1D shape function used for $\partial v_1/\partial x_1$,
`dv2dx[]`... values of the 1D shape function used for $\partial v_2/\partial x_2$.

Every quadrilateral/hexahedral Element should know whether it is distorted or not. If the element is not distorted, and if a volumetric product weak form is available, it should be called instead of the regular volumetric weak form on that element. If the element is distorted, then the regular volumetric form would be called.

For any weak form, the user would have to always provide the full 2D integration version first, and the product form would be optional.