Avoiding Some Numerical Quadrature on Higher-Order Quadrilateral Elements

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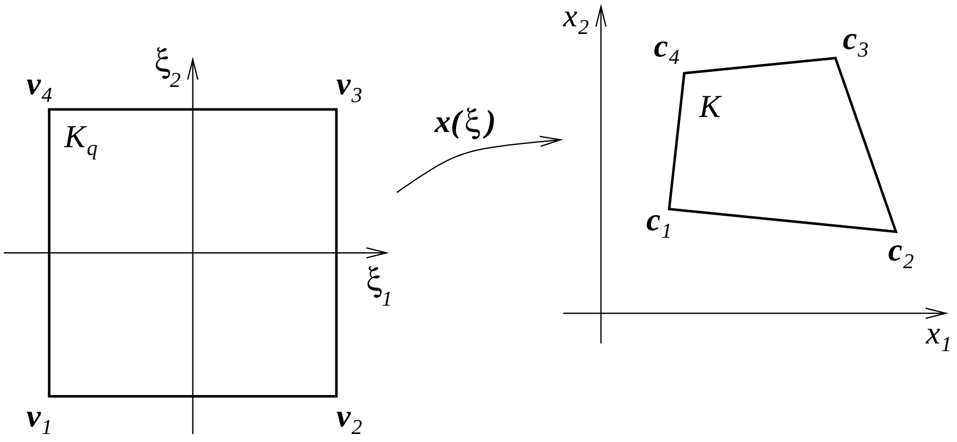
January 2009

Abstract

In higher-order finite element codes, numerical quadrature often is the most time-consuming part of stiffness matrix assembly. When equation coefficients are elementwise constant, the situation is simple on non-curved simplicial elements (triangles, tetrahedra). The number of precomputed integrals is $O(p^4)$ in 2D and $O(p^6)$ in 3D. In this paper we show that something similar can be done for L^2-products on non-curved quadrilateral elements when product shape functions are used. Here, the number of precomputed integrals only is $O(p^2)$. We also discuss integrals containing derivatives.

1 Reference map

Let us begin with introducing a sample reference domain K_q, a mesh quadrilateral K with straight edges, and a standard bilinear bijective reference map $\boldsymbol{x} : K_q \to K$.



Here, $\boldsymbol{c}_i = (c_i^1, c_i^2)$ are the physical coordinates of the vertices of K. We assume the same ordering of the vertices of K_q and K, i.e., $\boldsymbol{x}(\boldsymbol{v}_i) = \boldsymbol{c}_i$, $i = 1, 2, \ldots, 4$. To avoid sign issues in the substitution theorem, we also assume that the determinant of the Jacobi matrix of the map $\boldsymbol{x}(\boldsymbol{\xi})$ is positive everywhere in K_q.

The standard bilinear vertex functions ϕ_i are defined as follows [1]:

$$\begin{aligned}
\phi_1(\boldsymbol{\xi}) &= \frac{1}{4}(1-\xi_1)(1-\xi_2),\\
\phi_2(\boldsymbol{\xi}) &= \frac{1}{4}\xi_1(1-\xi_2),\\
\phi_3(\boldsymbol{\xi}) &= \frac{1}{4}\xi_1\xi_2,\\
\phi_4(\boldsymbol{\xi}) &= \frac{1}{4}(1-\xi_1)\xi_2.
\end{aligned}$$

Notice that $\phi_i(\boldsymbol{v}_j) = \delta_{ij}$. Recall from [1] that if the quadrilateral K does not have curved edges, the reference map $\boldsymbol{x}(\boldsymbol{\xi})$ has the form

$$\boldsymbol{x}(\boldsymbol{\xi}) = \sum_{i=1}^{4} \boldsymbol{c}_i\phi_i(\boldsymbol{\xi}) = \begin{pmatrix} c_1^1 \\ c_1^2 \end{pmatrix}\phi_1(\boldsymbol{\xi}) + \begin{pmatrix} c_2^1 \\ c_2^2 \end{pmatrix}\phi_2(\boldsymbol{\xi}) + \begin{pmatrix} c_3^1 \\ c_3^2 \end{pmatrix}\phi_3(\boldsymbol{\xi}) + \begin{pmatrix} c_4^1 \\ c_4^2 \end{pmatrix}\phi_4(\boldsymbol{\xi}).$$

2 Jacobi matrix

Partial derivatives of the shape functions have the form

$$\begin{aligned}
\frac{\partial}{\partial\xi_1}\phi_1(\boldsymbol{\xi}) &= -\frac{1}{4}(1-\xi_2),\ \frac{\partial}{\partial\xi_2}\phi_1(\boldsymbol{\xi}) = -\frac{1}{4}(1-\xi_1),\\
\frac{\partial}{\partial\xi_1}\phi_2(\boldsymbol{\xi}) &= \frac{1}{4}(1-\xi_2),\ \frac{\partial}{\partial\xi_2}\phi_2(\boldsymbol{\xi}) = -\frac{1}{4}\xi_1,\\
\frac{\partial}{\partial\xi_1}\phi_3(\boldsymbol{\xi}) &= \frac{1}{4}\xi_2,\ \frac{\partial}{\partial\xi_2}\phi_3(\boldsymbol{\xi}) = \frac{1}{4}\xi_1,\\
\frac{\partial}{\partial\xi_1}\phi_4(\boldsymbol{\xi}) &= -\frac{1}{4}\xi_2,\ \frac{\partial}{\partial\xi_2}\phi_4(\boldsymbol{\xi}) = \frac{1}{4}(1-\xi_1),
\end{aligned}$$

and thus the partial derivatives of the map $\boldsymbol{x}(\boldsymbol{\xi})$ are

$$\begin{aligned}
\frac{\partial x_1}{\partial\xi_1} &= \frac{1}{4}(-c_1^1+c_2^1) + \frac{\xi_2}{4}(c_1^1-c_2^1+c_3^1-c_4^1) = A^1 + C^1\xi_2,\\
\frac{\partial x_1}{\partial\xi_2} &= \frac{1}{4}(-c_1^1+c_4^1) + \frac{\xi_1}{4}(c_1^1-c_2^1+c_3^1-c_4^1) = B^1 + C^1\xi_2,\\
\frac{\partial x_2}{\partial\xi_1} &= \frac{1}{4}(-c_1^2+c_2^2) + \frac{\xi_2}{4}(c_1^2-c_2^2+c_3^2-c_4^2) = A^2 + C^2\xi_2,\\
\frac{\partial x_2}{\partial\xi_2} &= \frac{1}{4}(-c_1^2+c_4^2) + \frac{\xi_2}{4}(c_1^2-c_2^2+c_3^2-c_4^2) = B^2 + C^2\xi_2.
\end{aligned}$$

The Jacobian of the map $\boldsymbol{x}(\boldsymbol{\xi})$ has the form

$$
\begin{aligned}
\det\left(\frac{D\boldsymbol{x}}{D\boldsymbol{\xi}}\right) &= \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1} \\
&= (A^1 + C^1\xi_2)(B^2 + C^2\xi_2) - (B^1 + C^1\xi_2)(A^2 + C^2\xi_2) \\
&= D_0 + D_1\xi_1 + D_2\xi_2,
\end{aligned}
$$

where

$$
D_0 = A^1B^2 - B^1A^2, \quad D_1 = A^1C^2 - C^1A^2, \quad D_2 = C^1B^2 - B^1C^2.
$$

3 L^2 products

Let's calculate the product of two shape functions $\chi_i(\xi_1, \xi_2) = \varphi^1(\xi_1)\varphi^2(\xi_2)$ and $\chi_j(\xi_1, \xi_2) = \psi^1(\xi_1)\psi^2(\xi_2)$:

$$
\int_{K_q} \det\left(\frac{D\boldsymbol{x}}{D\boldsymbol{\xi}}\right) \chi_i(\xi_1, \xi_2)\chi_j(\xi_1, \xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2
$$

$$
= \int_{-1}^{1}\int_{-1}^{1} D_0\chi_i(\xi_1, \xi_2)\chi_j(\xi_1, \xi_2) + D_1\xi_1\chi_i(\xi_1, \xi_2)\chi_j(\xi_1, \xi_2) + D_2\xi_2\chi_i(\xi_1, \xi_2)\chi_j(\xi_1, \xi_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2
$$

$$
\begin{aligned}
&= D_0 \int_{-1}^{1} \varphi^1(\xi_1)\psi^1(\xi_1)\mathrm{d}\xi_1 \int_{-1}^{1} \varphi^2(\xi_2)\psi^2(\xi_2)\mathrm{d}\xi_2 \\
&+D_1 \int_{-1}^{1} \xi_1\varphi^1(\xi_1)\psi^1(\xi_1)\mathrm{d}\xi_1 \int_{-1}^{1} \varphi^2(\xi_2)\psi^2(\xi_2)\mathrm{d}\xi_2 \\
&+D_2 \int_{-1}^{1} \varphi^1(\xi_1)\psi^1(\xi_1)\mathrm{d}\xi_1 \int_{-1}^{1} \xi_2\varphi^2(\xi_2)\psi^2(\xi_2)\mathrm{d}\xi_2.
\end{aligned}
$$

Note that this integral can be assembled easily using two sets of precomputed one-dimensional integrals:

$$
\int_{-1}^{1} \varphi(\xi)\psi(\xi)\mathrm{d}\xi \quad \text{and} \quad \int_{-1}^{1} \xi\varphi(\xi)\psi(\xi)\mathrm{d}\xi.
$$

If the maximum polynomial order in the code is p, then this is just $2(p+1)^2$ one-dimensional integrals to store.

4 Products containing derivatives

Derivatives are transformed between the element K and the reference domain K_q as

$$
\nabla(\chi(\boldsymbol{x})) = \left(\frac{D\boldsymbol{x}}{D\boldsymbol{\xi}}\right)^{-T} \nabla_\xi(\chi \circ \boldsymbol{x}).
$$

see, e.g., [1]. The inverse Jacobi matrix of the map $\boldsymbol{x}(\boldsymbol{\xi})$ is

$$\left(\frac{D\boldsymbol{x}}{D\boldsymbol{\xi}}\right)^{-1} = \left(\begin{array}{cc} \dfrac{A^1 + C^1\xi_2}{D_0 + D_1\xi_1 + D_2\xi_2} & \dfrac{B^1 + C^1\xi_2}{D_0 + D_1\xi_1 + D_2\xi_2} \\ \dfrac{A^2 + C^2\xi_2}{D_0 + D_1\xi_1 + D_2\xi_2} & \dfrac{B^2 + C^2\xi_2}{D_0 + D_1\xi_1 + D_2\xi_2} \end{array}\right) = \left(\begin{array}{cc} j_{11}(\xi_1, \xi_2) & j_{12}(\xi_1, \xi_2) \\ j_{21}(\xi_1, \xi_2) & j_{22}(\xi_1, \xi_2) \end{array}\right)$$

which implies that we will not be able to split its entries into products of the form $\varphi(\xi_1)\psi(\xi_2)$. The only way to avoid wasteful repeated computation of integrals of the form

$$\int_{K_q} j_{mn}(\xi_1, \xi_2)\frac{\partial\varphi}{\partial\xi_r}\frac{\partial\psi}{\partial\xi_s}\mathrm{d}\xi_1\xi_2$$

would be to store them somehow, the full set for every element. Of course this is a huge amount of data, but on the other hand the integrals could be precalculated adaptively with a high degree of accuracy (more accurate than with maximum quadrature order).

This is a painful topic because it is so difficult and so important – any ideas are welcome.

References

[1] P. Solin, K. Segeth, I. Dolezel, *Higher-Order Finite Element Methods*, Chapman & Hall/CRC Press, Boca Raton, 2003.