Taylor-Galerkin Method
for Compressible Euler Equations

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1 Euler Equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0 \tag{1}$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \operatorname{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v}) + \nabla p = 0 \tag{2}$$

$$\frac{\partial \rho e}{\partial t} + \operatorname{div}((\rho e + p)\boldsymbol{v}) = 0 \tag{3}$$

Using the perfect gas state equation $p = \varrho RT$ with $R = c_p - c_v$, we obtain

$$p = (\kappa - 1)\left(\varrho e - \frac{1}{2}\varrho |\boldsymbol{v}|^2\right),$$

where $\kappa = c_p/c_v$.

2 Equations in conservative variables

The conservative variables are defined as follows,

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} \varrho \\ \varrho v_1 \\ \varrho v_2 \\ \varrho e \end{pmatrix},$$

where ϱ, v_1, v_2, e are the density, first velocity component, second velocity component, and total energy, respectively. In the conservation form, Euler equations can be written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial}{\partial x_1}\boldsymbol{f}_1(\boldsymbol{u}) + \frac{\partial}{\partial x_2}\boldsymbol{f}_2(\boldsymbol{u}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4}$$

where Eulerian fluxes have the following form

$$f_1(\boldsymbol{u}) = \begin{pmatrix} u_2 \\ \dfrac{u_2^2}{u_1} + (\kappa-1)(u_4 - \dfrac{u_2^2+u_3^2}{2u_1}) \\ \dfrac{u_3 u_2}{u_1} \\ \dfrac{u_2}{u_1}\left(\kappa u_4 - (\kappa-1)\dfrac{u_2^2+u_3^2}{2u_1}\right) \end{pmatrix}, \qquad f_2(\boldsymbol{u}) = \begin{pmatrix} u_3 \\ \dfrac{u_2 u_3}{u_1} \\ \dfrac{u_3^2}{u_1} + (\kappa-1)(u_4 - \dfrac{u_2^2+u_3^2}{2u_1}) \\ \dfrac{u_3}{u_1}\left(\kappa u_4 - (\kappa-1)\dfrac{u_2^2+u_3^2}{2u_1}\right) \end{pmatrix},$$

Equation (4) can be written as

$$\frac{\partial \boldsymbol{u}}{\partial t} + A_1(\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial x_1} + A_2(\boldsymbol{u})\frac{\partial \boldsymbol{u}}{\partial x_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5}$$

where the Jacobi matrices of the Eulerian fluxes $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ are defined as

$$A_1(\boldsymbol{u}) = D\boldsymbol{f}_1/D\boldsymbol{u}, \quad A_2(\boldsymbol{u}) = D\boldsymbol{f}_2/D\boldsymbol{u},$$

where Jacobi matrices $A_1(\boldsymbol{u})$ and $A_2(\boldsymbol{u})$ have the form

$$A_1(\boldsymbol{u}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \dfrac{(\kappa-1)(u_2^2+u_3^2)}{2u_1^2} - \dfrac{u_2^2}{u_1^2} & (3-\kappa)\dfrac{u_2}{u_1} & (1-\kappa)\dfrac{u_3}{u_1} & \kappa-1 \\ -\dfrac{u_2 u_3}{u_1^2} & \dfrac{u_3}{u_1} & \dfrac{u_2}{u_1} & 0 \\ \dfrac{u_2}{u_1}\left(\dfrac{(\kappa-1)(u_2^2+u_3^2)}{u_1^2} - \kappa\dfrac{u_4}{u_1}\right) & \kappa\dfrac{u_4}{u_1} - \dfrac{(\kappa-1)u_2^2}{u_1^2} - \dfrac{(\kappa-1)(u_2^2+u_3^2)}{2u_1^2} & \dfrac{(1-\kappa)u_2u_3}{u_1^2} & \kappa\dfrac{u_2}{u_1} \end{pmatrix}$$

$$A_2(\boldsymbol{u}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -\dfrac{u_2 u_3}{u_1^2} & \dfrac{u_3}{u_1} & \dfrac{u_2}{u_1} & 0 \\ \dfrac{(\kappa-1)(u_2^2+u_3^2)}{2u_1^2} - \dfrac{u_3^2}{u_1^2} & (1-\kappa)\dfrac{u_2}{u_1} & (3-\kappa)\dfrac{u_3}{u_1} & \kappa-1 \\ \dfrac{u_3}{u_1}\left(\dfrac{(\kappa-1)(u_2^2+u_3^2)}{u_1^2} - \kappa\dfrac{u_4}{u_1}\right) & \dfrac{(1-\kappa)u_2u_3}{u_1^2} & \kappa\dfrac{u_4}{u_1} - \dfrac{(\kappa-1)u_3^2}{u_1^2} - \dfrac{(\kappa-1)(u_2^2+u_3^2)}{2u_1^2} & \kappa\dfrac{u_3}{u_1} \end{pmatrix}$$

3 Taylor-Galerkin Method

Let us expand $\boldsymbol{u}$ in time using second-order Taylor series

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \Delta t\frac{\partial \boldsymbol{u}^n}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial}{\partial t}\frac{\partial \boldsymbol{u}^{n+1}}{\partial t} + O(\Delta t^3). \tag{6}$$

Now, plug in the equation (4) to the Taylor series (6)

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}}{\Delta t}+\left(\frac{\partial}{\partial x_1}\boldsymbol{f}_1(\boldsymbol{u}^n)+\frac{\partial}{\partial x_2}\boldsymbol{f}_2(\boldsymbol{u}^n)\right)+\frac{\Delta t}{2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x_1}\boldsymbol{f}_1(\boldsymbol{u}^{n+1})+\frac{\partial}{\partial x_2}\boldsymbol{f}_2(\boldsymbol{u}^{n+1})\right)=0 \tag{7}$$

We can rewrite the third term in (7) in terms of spatial derivatives using integration by parts

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x_1}\boldsymbol{f}_1(\boldsymbol{u})+\frac{\partial}{\partial x_2}\boldsymbol{f}_2(\boldsymbol{u})\right)=\frac{\partial}{\partial x_1}\frac{\partial}{\partial t}\boldsymbol{f}_1(\boldsymbol{u})+\frac{\partial}{\partial x_2}\frac{\partial}{\partial t}\boldsymbol{f}_2(\boldsymbol{u})=\frac{\partial}{\partial x_1}\frac{\partial \boldsymbol{f}_1}{\partial \boldsymbol{u}}\frac{\partial \boldsymbol{u}}{\partial t}+\frac{\partial}{\partial x_2}\frac{\partial \boldsymbol{f}_2}{\partial \boldsymbol{u}}\frac{\partial \boldsymbol{u}}{\partial t}=$$

$$=-\frac{\partial}{\partial x_1}A_1(\boldsymbol{u})\left(\frac{\partial}{\partial x_1}\boldsymbol{f}_1(\boldsymbol{u})+\frac{\partial}{\partial x_2}\boldsymbol{f}_2(\boldsymbol{u})\right)-\frac{\partial}{\partial x_2}A_2(\boldsymbol{u})\left(\frac{\partial}{\partial x_1}\boldsymbol{f}_1(\boldsymbol{u})+\frac{\partial}{\partial x_2}\boldsymbol{f}_2(\boldsymbol{u})\right) \tag{8}$$

Now, substituing (8) into (7) we obtain

$$\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}}{\Delta t}+\sum\frac{\partial}{\partial x_k}\boldsymbol{f}_k(\boldsymbol{u}^n)-\frac{\Delta t}{2}\sum\frac{\partial}{\partial x_k}A_k(\boldsymbol{u}^{n+1})\left(\frac{\partial}{\partial x_1}\boldsymbol{f}_1(\boldsymbol{u}^{n+1})+\frac{\partial}{\partial x_2}\boldsymbol{f}_2(\boldsymbol{u}^{n+1})\right)=0 \tag{9}$$

Proceeding with the Galerkin method, we multiply (9) by a test function $\boldsymbol{v}$ and integrate over the domain Ω to get the following scheme

$$\int_\Omega\frac{\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}}{\Delta t}\boldsymbol{v}-\int_\Omega\sum\boldsymbol{f}_k(\boldsymbol{u}^n)\frac{\partial \boldsymbol{v}}{\partial x_k}+\int_{\partial\Omega}\sum\boldsymbol{f}_k(\boldsymbol{u}^n)\boldsymbol{v}n_k+$$

$$+\frac{\Delta t}{2}\int_\Omega\sum A_k(\boldsymbol{u}^{n+1})\left(\frac{\partial}{\partial x_1}\boldsymbol{f}_1(\boldsymbol{u}^{n+1})+\frac{\partial}{\partial x_2}\boldsymbol{f}_2(\boldsymbol{u}^{n+1})\right)\frac{\partial \boldsymbol{v}}{\partial x_k}-$$

$$-\frac{\Delta t}{2}\int_{\partial\Omega}\sum A_k(\boldsymbol{u}^{n+1})\left(\frac{\partial}{\partial x_1}\boldsymbol{f}_1(\boldsymbol{u}^{n+1})+\frac{\partial}{\partial x_2}\boldsymbol{f}_2(\boldsymbol{u}^{n+1})\right)\boldsymbol{v}n_k=0 \tag{10}$$

or in simplified form

$$\frac{1}{\Delta t}\int_\Omega\boldsymbol{u}^{n+1}\boldsymbol{v}+\frac{\Delta t}{2}\int_\Omega\boldsymbol{A}(\boldsymbol{u}^{n+1})\mathrm{div}\boldsymbol{f}^{n+1}\ \cdot\nabla\boldsymbol{v}-\frac{\Delta t}{2}\int_{\partial\Omega}\boldsymbol{A}(\boldsymbol{u}^{n+1})\mathrm{div}\boldsymbol{f}^{n+1}\ \cdot\boldsymbol{n}\ \boldsymbol{v}=$$

$$=\frac{1}{\Delta t}\int_\Omega\boldsymbol{u}^{n}\boldsymbol{v}+\int_\Omega\boldsymbol{f}^n\cdot\nabla\boldsymbol{v}-\int_{\partial\Omega}\boldsymbol{f}^n\cdot\boldsymbol{n}\ \boldsymbol{v} \tag{11}$$

Now, using relations

$$\begin{aligned}\boldsymbol{f}&=\boldsymbol{A}\,\boldsymbol{u}\\ \mathrm{div}\boldsymbol{f}&=\boldsymbol{A}\cdot\nabla\boldsymbol{u}\end{aligned}$$

we can rewrite (11) in the following form

$$\frac{1}{\Delta t}\int_\Omega\boldsymbol{u}^{n+1}\boldsymbol{v}+\frac{\Delta t}{2}\int_\Omega\boldsymbol{A}^{n+1}(\boldsymbol{A}^{n+1}\cdot\nabla\boldsymbol{u}^{n+1})\ \cdot\nabla\boldsymbol{v}-\frac{\Delta t}{2}\int_{\partial\Omega}\boldsymbol{A}^{n+1}\ (\boldsymbol{A}^{n+1}\cdot\nabla\boldsymbol{u}^{n+1})\ \cdot\boldsymbol{n}\ \boldsymbol{v}=$$

$$=\frac{1}{\Delta t}\int_\Omega\boldsymbol{u}^{n}\boldsymbol{v}+\int_\Omega\boldsymbol{A}^n\boldsymbol{u}^n\cdot\nabla\boldsymbol{v}-\int_{\partial\Omega}\boldsymbol{A}^n\boldsymbol{u}^n\cdot\boldsymbol{n}\ \boldsymbol{v} \tag{12}$$

Let me rewrite (12) in detail for implementation purposes

$$\frac{1}{\Delta t}\int_\Omega \boldsymbol{u}^{n+1}\boldsymbol{v} +$$

$$\frac{\Delta t}{2}\int_\Omega A_1^{n+1}\left(A_1^{n+1}\frac{\partial \boldsymbol{u}^{n+1}}{\partial x}+A_2^{n+1}\frac{\partial \boldsymbol{u}^{n+1}}{\partial y}\right)\frac{\partial \boldsymbol{v}}{\partial x}+A_2^{n+1}\left(A_1^{n+1}\frac{\partial \boldsymbol{u}^{n+1}}{\partial x}+A_2^{n+1}\frac{\partial \boldsymbol{u}^{n+1}}{\partial y}\right)\frac{\partial \boldsymbol{v}}{\partial y}$$

$$-\frac{\Delta t}{2}\int_\Omega A_1^{n+1}\left(A_1^{n+1}\frac{\partial \boldsymbol{u}^{n+1}}{\partial x}+A_2^{n+1}\frac{\partial \boldsymbol{u}^{n+1}}{\partial y}\right)n_x\boldsymbol{v}+A_2^{n+1}\left(A_1^{n+1}\frac{\partial \boldsymbol{u}^{n+1}}{\partial x}+A_2^{n+1}\frac{\partial \boldsymbol{u}^{n+1}}{\partial y}\right)n_y\boldsymbol{v}$$

$$=\frac{1}{\Delta t}\int_\Omega \boldsymbol{u}^n\boldsymbol{v}+\int_\Omega\left(A_1^n\boldsymbol{u}^n\frac{\partial \boldsymbol{v}}{\partial x}+A_2^n\boldsymbol{u}^n\frac{\partial \boldsymbol{v}}{\partial y}\right)-\int_{\partial\Omega}\left(A_1^n\,\boldsymbol{u}^n\,n_x\boldsymbol{v}+A_2^n\,\boldsymbol{u}^n\,n_y\boldsymbol{v}\right)$$

Therefore 1-th equation is

$$\frac{1}{\Delta t}\int_\Omega u_1^{n+1}v_1$$

$$+\ \frac{\Delta t}{2}\int_\Omega \sum_j (A_1^{n+1})_{1j}\left(\sum_k (A_1^{n+1})_{jk}\frac{\partial u_k^{n+1}}{\partial x}+\sum_k (A_2^{n+1})_{jk}\frac{\partial u_k^{n+1}}{\partial y}\right)\frac{\partial v_1}{\partial x}$$

$$+\ \frac{\Delta t}{2}\int_\Omega \sum_j (A_2^{n+1})_{1j}\left(\sum_k (A_1^{n+1})_{jk}\frac{\partial u_k^{n+1}}{\partial x}+\sum_k (A_2^{n+1})_{jk}\frac{\partial u_k^{n+1}}{\partial y}\right)\frac{\partial v_1}{\partial y}$$

$$-\ \frac{\Delta t}{2}\int_{\partial\Omega} \sum_j (A_1^{n+1})_{1j}\left(\sum_k (A_1^{n+1})_{jk}\frac{\partial u_k^{n+1}}{\partial x}+\sum_k (A_2^{n+1})_{jk}\frac{\partial u_k^{n+1}}{\partial y}\right)n_x\ v_1$$

$$-\ \frac{\Delta t}{2}\int_{\partial\Omega} \sum_j (A_2^{n+1})_{1j}\left(\sum_k (A_1^{n+1})_{jk}\frac{\partial u_k^{n+1}}{\partial x}+\sum_k (A_2^{n+1})_{jk}\frac{\partial u_k^{n+1}}{\partial y}\right)n_y\ v_1$$

$$=\frac{1}{\Delta t}\int_\Omega u_1^n v_1+\int_\Omega\left(\sum_k (A_1^n)_{1k}\ u_k^n\frac{\partial v_1}{\partial x}+\sum_k (A_2^n)_{1k}\ u_k^n\frac{\partial v_1}{\partial y}\right)$$

$$-\ \int_{\partial\Omega}\left(\sum_k (A_1^n)_{1k}\ u_k^n n_x v_1+\sum_k (A_2^n)_{1k}\ u_k^n n_y v_1\right)$$

Separated into 4 parts that correspond to blocks in stiffness matrix (superscript $n+1$ is omitted)

$$\frac{1}{\Delta t}\int_\Omega u_1 v_1$$

$$+\frac{\Delta t}{2}\int_\Omega \sum_j (A_1)_{1j}\left((A_1)_{j1}\frac{\partial u_1}{\partial x}+(A_2)_{j1}\frac{\partial u_1}{\partial y}\right)\frac{\partial v_1}{\partial x}+\sum_j (A_2)_{1j}\left((A_1)_{j1}\frac{\partial u_1}{\partial x}+(A_2)_{j1}\frac{\partial u_1}{\partial y}\right)\frac{\partial v_1}{\partial y}$$

$$-\frac{\Delta t}{2}\int_{\partial\Omega}\left[\sum_j (A_1)_{1j}\left((A_1)_{j1}\frac{\partial u_1}{\partial x}+(A_2)_{j1}\frac{\partial u_1}{\partial y}\right)n_x v_1+\sum_j (A_2)_{1j}\left((A_1)_{j1}\frac{\partial u_1}{\partial x}+(A_2)_{j1}\frac{\partial u_1}{\partial y}\right)n_y v_1\right]$$

$$+\frac{\Delta t}{2}\int_\Omega \sum_j (A_1)_{1j}\left((A_1)_{j2}\frac{\partial u_2}{\partial x}+(A_2)_{j2}\frac{\partial u_2}{\partial y}\right)\frac{\partial v_1}{\partial x}+\sum_j (A_2)_{1j}\left((A_1)_{j2}\frac{\partial u_2}{\partial x}+(A_2)_{j2}\frac{\partial u_2}{\partial y}\right)\frac{\partial v_1}{\partial y}$$

$$-\frac{\Delta t}{2}\int_{\partial\Omega}\left[\sum_j (A_1)_{1j}\left((A_1)_{j2}\frac{\partial u_2}{\partial x}+(A_2)_{j2}\frac{\partial u_2}{\partial y}\right)n_x v_1+\sum_j (A_2)_{1j}\left((A_1)_{j2}\frac{\partial u_2}{\partial x}+(A_2)_{j2}\frac{\partial u_2}{\partial y}\right)n_y v_1\right]$$

$$+\frac{\Delta t}{2}\int_\Omega \sum_j (A_1)_{1j}\left((A_1)_{j3}\frac{\partial u_3}{\partial x}+(A_2)_{j3}\frac{\partial u_3}{\partial y}\right)\frac{\partial v_1}{\partial x}+\sum_j (A_2)_{1j}\left((A_1)_{j3}\frac{\partial u_3}{\partial x}+(A_2)_{j3}\frac{\partial u_3}{\partial y}\right)\frac{\partial v_1}{\partial y}$$

$$-\frac{\Delta t}{2}\int_{\partial\Omega}\left[\sum_j (A_1)_{1j}\left((A_1)_{j3}\frac{\partial u_3}{\partial x}+(A_2)_{j3}\frac{\partial u_3}{\partial y}\right)n_x v_1+\sum_j (A_2)_{1j}\left((A_1)_{j3}\frac{\partial u_3}{\partial x}+(A_2)_{j3}\frac{\partial u_3}{\partial y}\right)n_y v_1\right]$$

$$+\frac{\Delta t}{2}\int_\Omega \sum_j (A_1)_{1j}\left((A_1)_{j4}\frac{\partial u_4}{\partial x}+(A_2)_{j4}\frac{\partial u_4}{\partial y}\right)\frac{\partial v_1}{\partial x}+\sum_j (A_2)_{1j}\left((A_1)_{j4}\frac{\partial u_4}{\partial x}+(A_2)_{j4}\frac{\partial u_4}{\partial y}\right)\frac{\partial v_1}{\partial y}$$

$$-\frac{\Delta t}{2}\int_{\partial\Omega}\left[\sum_j (A_1)_{1j}\left((A_1)_{j4}\frac{\partial u_4}{\partial x}+(A_2)_{j4}\frac{\partial u_4}{\partial y}\right)n_x v_1+\sum_j (A_2)_{1j}\left((A_1)_{j4}\frac{\partial u_4}{\partial x}+(A_2)_{j4}\frac{\partial u_4}{\partial y}\right)n_y v_1\right]$$

$$=\frac{1}{\Delta t}\int_\Omega u_1^n v_1+\int_\Omega\left(\sum_k (A_1^n)_{1k}\; u_k^n\frac{\partial v_1}{\partial x}+\sum_k (A_2^n)_{1k}\; u_k^n\frac{\partial v_1}{\partial y}\right)$$

$$-\int_{\partial\Omega}\left(\sum_k (A_1^n)_{1k}\; u_k^n n_x v_1+\sum_k (A_2^n)_{1k}\; u_k^n n_y v_1\right)$$