

TRILINOS Tutorial

Heat equation:

$$\rho c_p \frac{\partial T}{\partial t} - \nabla \cdot (k \nabla T) - \dot{Q} = 0, \quad (x, y) \in \Omega$$

Essential boundary conditions

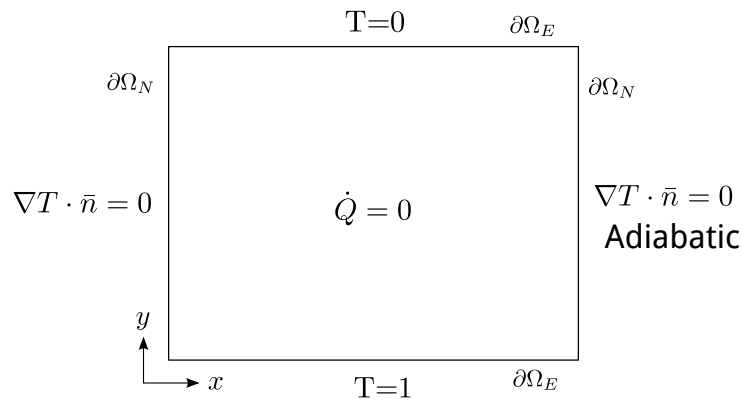
$$T(x, y) = \alpha(x, y) \quad (x, y) \in \partial\Omega_E$$

Natural boundary conditions

$$k(x, y) \frac{\partial T}{\partial \bar{n}} = k \nabla T \cdot \bar{n} = \beta(x, y) \quad (x, y) \in \partial\Omega_N$$

Simpler problem for the sake of illustration:

steady state head conduction in a rectangle, no sources or sinks



$$\nabla \cdot (k \nabla T) = 0$$

$$T(x, 0) = 1, T(x, 1) = 0$$

Galerkin: Multiply by test function v and integrate over Ω .

$$\iint_{\Omega} [\nabla \cdot (k \nabla T)] v \, dx dy = 0$$

Product rule for ∇ :

$$\nabla \cdot (v k \nabla T) = (\nabla v) \cdot (k \nabla T) + v \nabla \cdot (k \nabla T)$$

$$v \nabla \cdot (k \nabla T) = \nabla \cdot (v k \nabla T) - (\nabla v) \cdot (k \nabla T)$$

$$\iint_{\Omega} [\nabla \cdot (k \nabla T)] v \, dx dy = \iint_{\Omega} \nabla \cdot (v k \nabla T) \, dx dy - \iint_{\Omega} (\nabla v) \cdot (k \nabla T) \, dx dy$$

Divergence theorem:

$$\iint_{\Omega} \nabla \cdot a \, dx dy = \int_{\partial\Omega} a \cdot \bar{n} dS$$

So

$$\iint_{\Omega} \nabla \cdot (vk\nabla T) \, dx dy = \int_{\partial\Omega} vk\nabla T \cdot \bar{n} \, dS$$

We now have

$$\iint_{\Omega} \nabla v \cdot k\nabla T \, dx dy - \int_{\partial\Omega} vk\nabla T \cdot \bar{n} \, dS = 0$$

$$\nabla T \cdot \bar{n} = 0 \text{ on } \partial\Omega_N$$

Finite Element Problem

$$\text{Let } T(x, y) = u(x, y) \approx U(x, y) = \sum_{j=1}^N c_j \phi_j(x, y)$$

$$v(x, y) \approx V(x, y) = \sum_{j=1}^N d_j \phi_j(x, y)$$

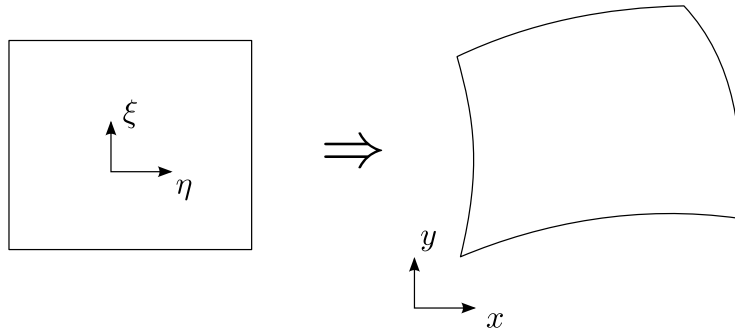
where ϕ_j are FE basis functions.

$$\iint \nabla v \cdot k\nabla T \, dx dy = \sum_E \iint_{\Omega_e} \nabla V \cdot k\nabla U \, dx dy = 0$$

$$\nabla V \cdot \nabla U = v_x u_x + v_y u_y$$

$$\iint_{\Omega_e} [k(V_x U_x + V_y U_y)] \, dx dy$$

Canonical Element Transformation



$$\text{Let } U_0 = U(x(\xi, \eta), y(\xi, \eta))$$

$$V_0 = V(x(\xi, \eta), y(\xi, \eta))$$

$$J = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \quad \det J = x_\xi y_\eta - x_\eta y_\xi$$

$$V_x = \frac{\partial V_0}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial V_0}{\partial \eta} \frac{\partial \eta}{\partial x} = V_{0,\eta} \eta_x + V_{0,\xi} \xi_x$$

$$\iint_{\Omega_0} [k(V_{0,\xi} \xi_x + V_{0,\eta} \eta_x)(U_{0,\xi} \xi_x + U_{0,\eta} \eta_x) + k(V_{0,\xi} \xi_y + V_{0,\eta} \eta_y)(U_{0,\xi} \xi_y + U_{0,\eta} \eta_y)] \det J_e \, d\xi d\eta$$

$$\iint_{\Omega_0} [k(\xi_x^2 + \xi_y^2) V_{0,\xi} U_{0,\xi} + k(\xi_x \eta_x + \xi_y \eta_y)(V_{0,\xi} U_{0,\eta} + V_{0,\eta} U_{0,\xi}) + k(\eta_x^2 + \eta_y^2) V_{0,\eta} U_{0,\eta}] \det J_e \, d\xi d\eta$$

$$\text{Let } U_0(\xi, \eta) = c_e^T N(\xi, \eta) = N^T(\xi, \eta) c_e$$

$$V_0(\xi, \eta) = d_e^T N(\xi, \eta) = N^T(\xi, \eta) d_e$$

$$\begin{aligned} &= \iint_{\Omega_0} [\quad k \quad (\xi_x^2 + \xi_y^2) d_e^T N_\xi N_\xi^T c_e + \\ &\quad k \quad (\xi_x \eta_x + \xi_y \eta_y) [d_e^T N_\xi N_\eta^T c_e + d_e^T N_\eta N_\xi^T c_e] + \\ &\quad k \quad (\eta_x^2 + \eta_y^2) d_e^T N_\eta N_\eta^T c_e] \det J_e \, d\xi d\eta = d_e^T K_e c_e \end{aligned}$$

$$K_e = \iint_{\Omega_0} [k(\xi_x^2 + \xi_y^2) N_\xi N_\xi^T + k(\xi_x \eta_x + \xi_y \eta_y) [N_\xi N_\eta^T + N_\eta N_\xi^T] + k(\eta_x^2 + \eta_y^2) N_\eta N_\eta^T] \det J_e \, d\xi d\eta$$

$$\sum_E d_e^T K_e c_e = 0$$

$$KC = 0$$