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ON *hp*-FEM FOR SINGULAR ELECTRO- AND MAGNETOSTATICS PROBLEMS

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Abstract - In this paper we explain the origin of singularities in electro- and magnetostatics problems, and mention the major difficulties related to their efficient and accurate numerical resolution. We give the basic ideas of the *hp*-FEM, and describe briefly our modular object-oriented *hp*-FEM system HERMES. Superiority of the *hp*-FEM over the standard FEM for problems with singularities is demonstrated on a practical electrostatics simulation.

1. Introduction

The problems of electro- and magnetostatics are described by means of elliptic partial differential equations (PDEs) that are obtained from the Maxwell's equations of electromagnetics by introducing suitable scalar potentials and applying appropriate constitutive relations. The elliptic PDEs are known to be well-posed and to have unique solutions under reasonable assumptions on the coefficients and data. Their accurate numerical solution, however, becomes challenging when the computational domain contains sharp re-entrant corners, cusps or edges, where the electric and/or magnetic fields typically are singular.

In the recent years we witness a rapidly increasing number of such problem in computational engineering (design of various electrostatic and electromagnetic appliances including, e.g., electrostatic micromotors, computer manufacturing, etc.). The best known numerical method for problems with singularities is the *hp*-FEM, which is a sophisticated version of the finite element method (FEM) that uses elements of variable size (h) and polynomial degree (p) in order to attain the maximum possible accuracy with fewest degrees of freedom (DOFs). The extremely fast (exponential) convergence of the *hp*-FEM for elliptic problems was discovered by I. Babuška et al (see [1] – [3]). By *exponential convergence* we mean that the error vanishes exponentially with the number of DOFs in the discrete problem. Standard FEM have much slower (polynomial) convergence rates.

Despite its great computational potential and solid mathematical foundation, though, the *hp*-FEM has not yet become a standard tool in computational engineering and science. The main reason is that there still remain numerous challenging open problems associated with practical computational aspects, such as the construction of optimal higher-order shape functions in higher spatial dimensions, design of suitable data structures and efficient scalable algorithms to accommodate nontrivial tasks related to *hp*-FEM assembling algorithms, robust PDE-independent error estimation, automatic *hp*-adaptivity and others. These tasks require expertise both in mathematics and computer science. It is the goal of our interdisciplinary team consisting of mathematicians, computer scientists and electrical engineers to develop the *hp*-FEM and present its convincing applications in such a way that it becomes a standard tool in the engineering community.

In this paper, after briefly recalling the elliptic PDEs governing electro- and magnetostatics in Section 2, the basic ideas of the *hp*-FEM are summarized in Section 3. Main features of our **HiER**archic **M**odular finite **E**lement **S**ystem (HERMES) are described in Section 4. A numerical example comparing the performances of the standard (piecewise-linear) FEM and the *hp*-FEM is presented in Section 5.

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2. Governing Equations and Computational Challenges

Putting together the Gauss' law for electricity $\nabla \cdot \boldsymbol{D} = \varrho$, the constitutive relation $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and the gradient expression $\boldsymbol{E} = -\nabla(\varphi_e + C)$ for stationary electric field, we obtain a second-order elliptic PDE of the form

$$-\nabla \cdot (\epsilon \nabla \varphi_e) = \varrho. \tag{2.1}$$

Analogously, the Gauss law for magnetism $\nabla \cdot \boldsymbol{B} = 0$ together with the constitutive relation $\boldsymbol{B} = \mu \boldsymbol{H}$ and the gradient expression $\boldsymbol{H} = -\nabla(\varphi_m + C)$ yield a second-order elliptic equation

$$-\nabla \cdot (\mu \nabla \varphi_m) = 0. \tag{2.2}$$

There are two basic challenges related to the numerical solution of the equations (2.1) and (2.2):

- at re-entrant corners the potentials φ_e and φ_m have a singular gradient (which, with a minus sign, is the electric field $\boldsymbol{E}$),
- on material interfaces with large jumps of the material parameters the electric field $\boldsymbol{E}$ is discontinuous.

Both these phenomena are shown in Fig. 1, which is related to a problem that will be described in Section 5.

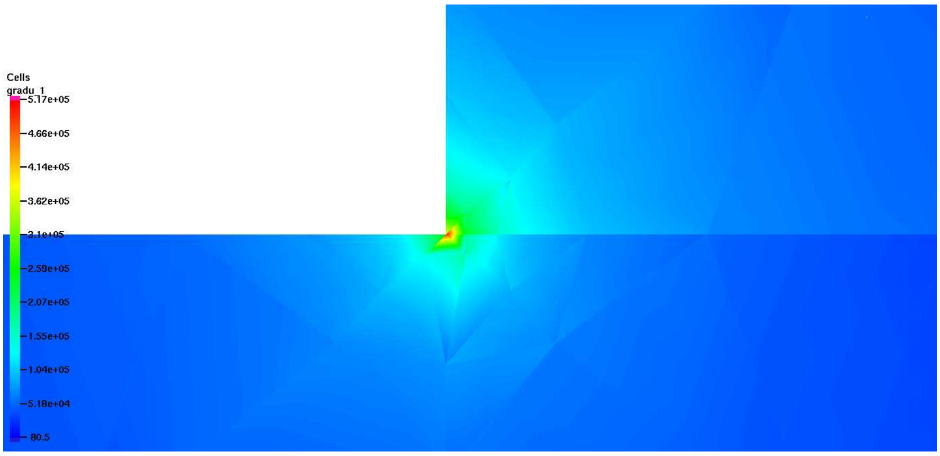

Figure 1: Singularity of $|\mathbf{E}|$ at a re-entrant corner and discontinuity along a horizontal material interface starting at the corner **(zoom = 1000)**.

Fig. 1 illustrates that the singularities of the electric field $\boldsymbol{E}$ occur at extremely small scales – typically on the level of $10^{-3} - 10^{-6}$ of a characteristic length of the problem. Because of that, standard piecewise-linear FEM usually are not able to capture the singular behavior of the electric field $\boldsymbol{E}$ sufficiently. However, due to the extremely large values of $\boldsymbol{E}$ at the singularities, a failure to resolve them sufficiently results into a large error,

$$\|\boldsymbol{e}_{h,p}\|_{H^1(\Omega)} = \left(\int_\Omega {\boldsymbol{e}_{h,p}}^2 + |\nabla \boldsymbol{e}_{h,p}|^2 \, \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{2}}, \tag{2.3}$$

where $\boldsymbol{e}_{h,p} = \varphi - \varphi_{h,p}$ is the difference of the exact and approximate solutions. The norm of error used in (2.3) is called the H^1-norm – this is the appropriate mathematical measure for the error of second-order elliptic PDEs including (2.1) and (2.2).

3. Basic Ideas of the *hp*-FEM

Currently the best known numerical method for singular problems in electro- and magnetostatics and electromagnetics is the *hp*-FEM. As we said earlier, this is a sophisticated version of the FEM that combines elements of variable size and variable polynomial degree to minimize the error most efficiently. In order to achieve a non-uniform distribution of the polynomial degree in the finite element mesh, one uses special *hierarchic* shape functions. On triangular elements, the hierarchic shape functions comprise *vertex functions* that represent the solution at the vertices (Fig.4), *edge functions* that represent the solution on the edges (Fig. 5), and *bubble functions* that represent the solution in the element interior (Fig. 6).

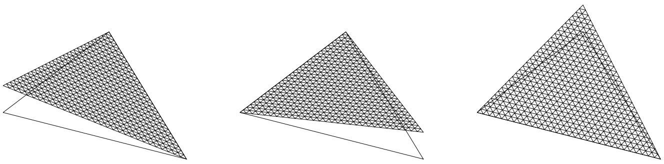

Figure 2: Vertex functions.

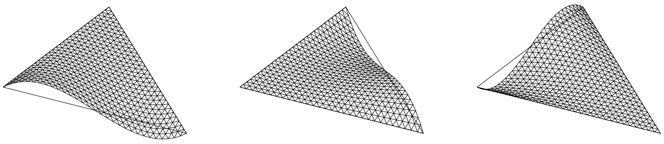

Figure 3: Cubic edge functions.

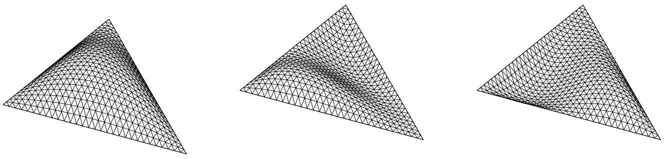

Figure 4: Cubic bubble function and fourth-order bubble functions.

An *hp*-FEM solver has to distribute automatically large high-degree elements where the solution is very smooth, and small low-degree elements where the solution is singular, oscillates or has internal or boundary layers. The automation of this process is highly nontrivial and it involves a lot of ongoing research activity. For more details we refer the reader to to [4, 7] and the references therein.

4. The Modular *hp*-FEM System HERMES

In order to facilitate the research in the *hp*-FEM and to ease its portability to practical engineering problems, we are developing a **HiER**archic **M**odular object-oriented multi-physics finite **E**lement **S**ystem (HERMES), whose structure is illustrated in Fig. 5. The system consists of a central *hp*-FEM module that also is capable of performing standard FEM computations as a special case of the *hp*-FEM. This module contains the PDE-independent *hp*-FEM technology such as the *hp*-FEM mesh preprocessing algorithms, connectivity and orientation algorithms, assembling procedures, error estimation procedures, mesh refinement algorithms, etc. The *hp*-FEM module communicates with small object classes representing PDE-dependent data including various types of finite elements, such as the standard continuous elements for elliptic problems and edge elements for the Maxwell's equations (already available), and the Taylor-Hood elements for the incompressible Navier-Stokes equations and Argyris elements for plate-bending problems (implementation in progress). The *hp*-FEM kernel communicates with various linear and nonlinear algebraic equation solvers, such as Trilinos (developed at Sandia National Labs), PETSc (developed at Livermore National Labs) and UMFPACK (developed at the University of Florida). This communication occurs through a universal sparse matrix interface of our provenience called *sMatrix*.

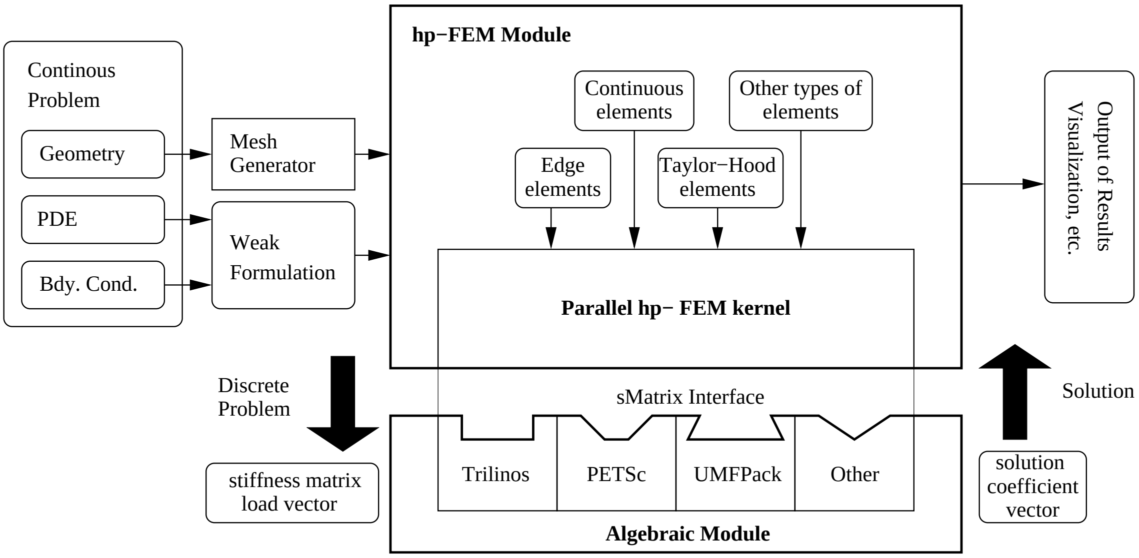

Figure 5: Modular structure of the FEM and *hp*-FEM system HERMES.

5. An Example Application of the *hp*-FEM

In order to illustrate the superiority of the *hp*-FEM over the standard FEM for problems with singularities, we solve a practical engineering task. The goal of the computation is to obtain the distribution of the electric field $\boldsymbol{E}$ induced by an insulated conductor in the vicinity of a point where the conductor leaves the wall. The computational domain Ω corresponding to this axisymmetric problem is depicted in Fig. 6.

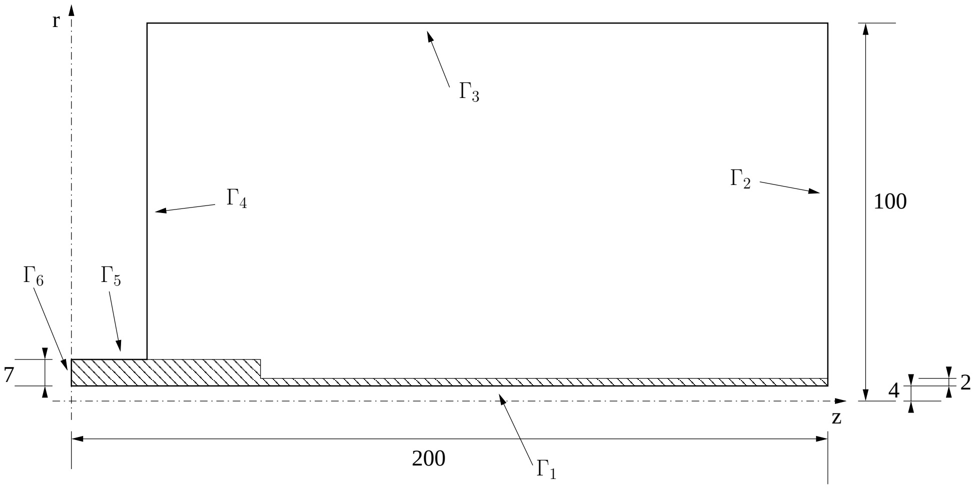

Figure 6: Computational domain (all measures are in millimeters).

The wall itself, where we are not interested in the solution, is not included in the domain Ω. The same holds for the conductor at the horizontal axis of symmetry. Both the wall and the conductor are handled via suitable boundary conditions (to be defined below). The hatched subdomain $\Omega_2 \subset \Omega$ represents the insulator with the relative permittivity $\epsilon_r = 10$. The relative permittivity in the rest of the domain is $\epsilon_r = 1$. This problem is more difficult compared to the previous one, because in addition to a re-entrant corner there is a material interface in the domain along which the electric field $\boldsymbol{E}$ is discontinuous (i.e., across which the scalar potential φ has a significant jump in the derivative). Solved is the standard potential equation of electrostatics (2.1) in cylindrical coordinates, equipped with the following boundary conditions: $\varphi = 220$ V on Γ_1, $\varphi = 0$ V on $\Gamma_4 \cup \Gamma_5$, and $\frac{\partial \varphi}{\partial \boldsymbol{\nu}} = 0$ on $\Gamma_2 \cup \Gamma_3 \cup \Gamma_6$.

We compare the results obtained by means of the piecewise-linear FEM and *hp*-FEM. The solution, gradient of the solution and the *hp*-mesh are shown in Figs. 7 – 8. In order to obtain the same level of accuracy, for the piecewise-linear FEM the same mesh was uniformly refined so that each edge was split into 23 subedges. An efficiency comparison is shown in Table 1.

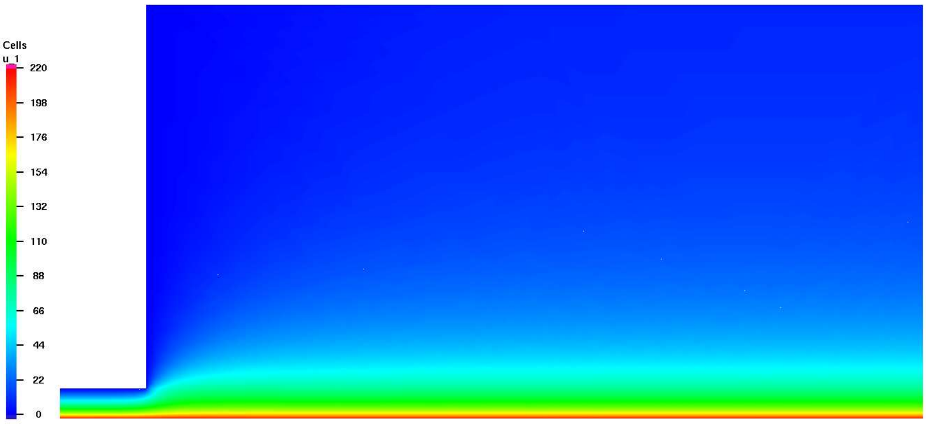

Figure 7: Solution of the insulator problem (electric potential φ).

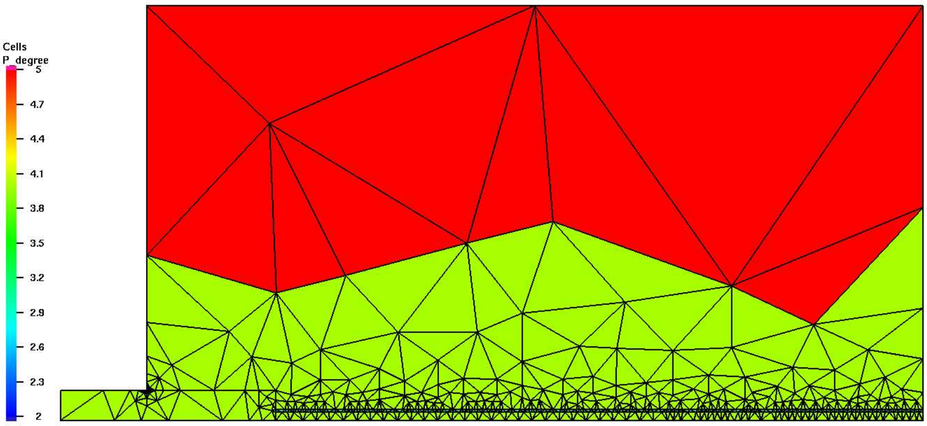

Figure 8: The *hp*-mesh. Large fifth-order elements are used far from the singularity and material interface, and very small quadratic elements are placed close to the re-entrant corner and the material interface

Table 1: Comparison of the number of degrees of freedom (DOF), relative error in the H^1-norm, number of iterations of the matrix solver and the CPU-time of the computation.

	linear FEM	*hp*-FEM
DOF	259393	6331
Error	1.617 %	1.521 %
Iterations	228	60
CPU time	**34 min.**	**11.58 sec.**

6. Current Research and Outlook

Our current research is devoted to the investigation of numerous theoretical and practical aspects of the *hp*-FEM, and its application to challenging engineering problems where the standard low-order methods such as the finite differences (FDM) or piecewise-linear FEM fail because of their low convergence rates. Typically, this is the case with challenging singular or multiscale problems in electrostatics and electromagnetics.

On the practical side we continue developing the modular multi-physics FEM and *hp*-FEM system HERMES that was introduced in Section 4. The first version of the 3D *hp*-FEM module will be finished and parallelized in Fall 2005. We further deal with numerous exciting open problems related to the optimization of automatic *hp*-adaptive algorithms and reference-solutions-based a-posteriori error estimators, parallelization of the codes, etc.

On the theoretical side we recently proved a discrete nonnegativity preservation principle for the *hp*-FEM in one spatial dimension [8], which was an open problem since 1981, and it is our next goal to extend this result to 2D and 3D. Further we investigate various ways to descrease the size and improve the conditioning of the global mass and stiffness matrices resulting from the *hp*-FEM discretizations [9, 10].

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