

Finite Element Analysis of a Convection-Diffusion Equation

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- 2 C-D Equation
- 3 Search for Optimal Mesh
- 4 Equidistributed Error Mesh (EE-mesh)
- 5 Comparisons
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Motivation

- Navier-Stokes equations (nonlinear, complicated),
- linear convection-diffusion equation (linear, simpler),
- goal: find optimal finite element mesh to resolve flow in boundary layers,
- aerospace, air force, naval research.

Model Problem

- Model Equation.

$$-\epsilon u'' - bu' = f \quad \text{in } \Omega = (0, 1),$$

$b > 0$ is a constant and $\epsilon > 0$,
 $\epsilon = 10^{-3}, 10^{-4}, \dots, 10^{-8}$.

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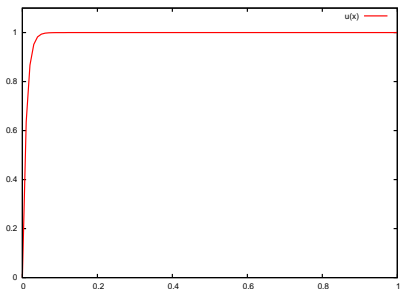
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- Boundary Conditions

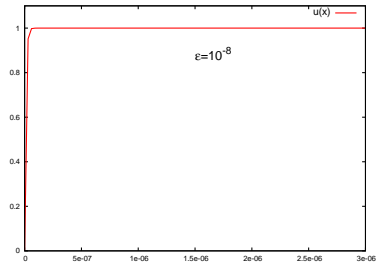
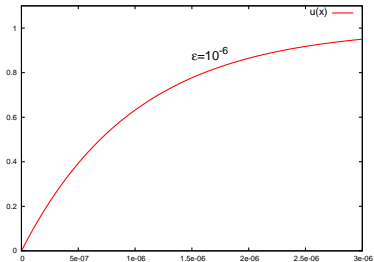
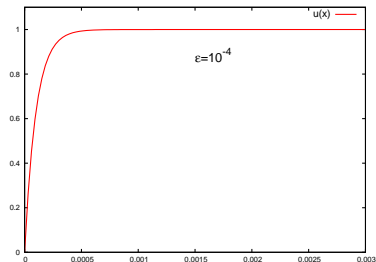
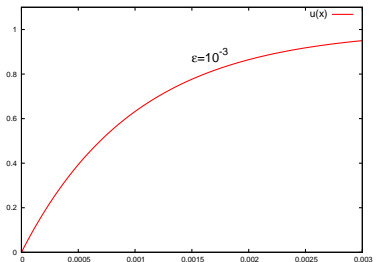
$$u(0) = 0, \quad u(1) = 1.$$

Exact Solution

- Consider the last model equation, with $\epsilon = 10^{-2}$, $b = 1$, $f(x) = 0$, the next graph show the exact solution.



Detail of the Boundary Layer



Weak Formulation

- Find a function $U \in V = H_0^1(\Omega)$ such that

$$a(U, v) = l(v) \quad \text{for all } v \in V, \quad G \in H^1(\Omega).$$

Weak Formulation

- Find a function $U \in V = H_0^1(\Omega)$ such that

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- The bilinear form $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$, is defined as

$$a(U, v) = \int_{\Omega} (\epsilon U' v' - b U' v) dx, \quad v \in V.$$

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- Linear form $l \in V'$, reads

$$l(v) = \int_{\Omega} (fv + \epsilon G' v' + b G' v) dx, \quad v \in V.$$

Existence and Uniqueness of Solution

- **Lemma (Lax-Milgram).** Let V be a Hilbert Space, $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$ a bounded V -elliptic bilinear form and $l \in V'$. Then there exists a unique solution the problem

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- Boundedness: $C = \max(\epsilon, b)$, using triangle inequality,

$$|a(U, v)| \leq \int_{\Omega} (\epsilon |U'v'| + b |U'v|) dx \leq C \int_{\Omega} (|U'v'| + |U'v|) dx.$$

Existence and Uniqueness of Solution

- Since $U', v' \in L^2(\Omega)$, the Hölder Inequality yields

$$\int_{\Omega} |U'v'| \, dx \leq \left(\int_{\Omega} |U'|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |v'|^2 \, dx \right)^{\frac{1}{2}} = |U|_{1,2} |v|_{1,2}.$$

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- We know $\|\cdot\|_{1,2}^2 = |\cdot|_{1,2}^2 + \|\cdot\|_2^2$, therefore

$$|U|_{1,2} |v|_{1,2} \leq \|U\|_{1,2} \|v\|_{1,2}, \quad |U|_{1,2} \|v\|_2 \leq \|U\|_{1,2} \|v\|_{1,2},$$

$$|a(U, v)| \leq 2C \|U\|_{1,2} \|v\|_{1,2}.$$

Existence and Uniqueness of Solution

- V-ellipticity: We split the expression $bU'v$ into two halves,

$$a(U, v) = \int_{\Omega} \epsilon U' v' dx - \int_{\Omega} \frac{b}{2} U' v dx - \int_{\Omega} \frac{b}{2} U' v dx.$$

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- Apply the Green's Theorem to one of these halves,

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- Substituting $U:=v$, we obtain

$$a(v, v) = \int_{\Omega} \epsilon (v')^2 dx + \int_{\Omega} \frac{b}{2} v v' dx - \int_{\Omega} \frac{b}{2} v' v dx = \int_{\Omega} \epsilon (v')^2 dx.$$

Existence and Uniqueness of Solution

- By Poincaré-Friedrichs' inequality, one obtains

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- Since the bilinear form $a(\cdot, \cdot)$ is bounded and V-elliptic, the Lax-Milgram lemma yields the existence and uniqueness of solution for every $f \in L^2(\Omega)$.

Finite Element Discretization

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- The elements K_m carry arbitrary polynomial degrees $1 \leq p_m, m = 1, 2, \dots, M$.
- For every element K_m , we define an affine reference map of the form:

$$x_{K_m}(\xi) = c_1^{(m)} + c_2^{(m)}\xi,$$

where

$$c_1^{(m)} = \frac{x_{m-1} + x_m}{2}, \quad c_2^{(m)} = \frac{x_m - x_{m-1}}{2}.$$

Finite Element Discretization

- The space $V_{h,p}$ corresponding to the mesh $\mathcal{T}_{h,p}$ has the form

$$V_{h,p} = \{v \in V; v|_{K_m} \in P^{p_m}(K_m) \text{ for all } m = 1, 2, \dots, M\}.$$

Equivalently,

$$V_{h,p} = \{v \in V; v|_{K_m \circ X_{K_m}} \in P^{p_m}(K_a) \text{ for all } m = 1, 2, \dots, M\}.$$

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- $K_a = (-1, 1)$, and $(f \circ g)(x) \equiv f(g(x))$.
- The dimension N of the space $V_{h,p}$ (number of DOF).

$$N = \dim(V_{h,p}) = M - 1 + \sum_{m=1}^M (p_m - 1) = -1 + \sum_{m=1}^M p_m.$$

Discrete Problem

- The discrete problem reads: Find a function $U_{h,p} \in V_{h,p}$ such that

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- We obtain

$$\sum_{j=1}^N y_j a(v_j, v_i) = l(v_i) \quad i = 1, 2, \dots, N.$$

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- Unknown coefficient vector \mathbf{Y}

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Dirichlet Boundary Conditions

- The nonhomogeneous Dirichlet boundary condition.

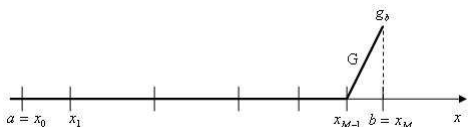


Figure: Piecewise-affine Dirichlet lift G . $G(0) = 0$, $G(1) = 1$.

$$(G \circ x_{K_m})(\xi) = \begin{cases} 0, & 1 \leq m < M, \\ g_b h_1(\xi), & m = M. \end{cases}$$

$$h_1(\xi) = \frac{1 + \xi}{2} \text{ is a Lobatto function.}$$

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- The exact solution is given by

$$u(x) = \frac{e^{-\frac{x}{\epsilon}} - 1}{e^{-\frac{1}{\epsilon}} - 1}.$$

Finite Computer Arithmetic

- Subtracting a small value from 1:

$$1 - 0.1 = 0.9$$

$$1 - 0.01 = 0.99$$

$$1 - 0.001 = 0.999$$

$$\vdots$$

$$1 - \underbrace{0.0\dots01}_{15} = 0.\underbrace{9\dots9}_{15}$$

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- After 16 digits of precision in the last example, we obtain 1.

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- We have $e^{-34} = 1.71391 \times 10^{-15}$ and $e^{-35} = 6.30512 \times 10^{-16}$.
- Let $s \in \mathbb{R}$, for $s \geq 35$, we have $e^{-s} < 10^{-15}$.

Numerical Approximation

- Consider $\epsilon = 10^{-3-k}$ with a nonnegative integer k . Then the exact solution

$$u(x) = \frac{e^{-\frac{x}{\epsilon}} - 1}{e^{-\frac{1}{\epsilon}} - 1} = \frac{e^{-\frac{x}{\epsilon}} - 1}{e^{-10^{3+k}} - 1}.$$

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- Analogously,

$$u(x) \approx 1$$

for all $x \in [35\epsilon, 1]$.

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- **Stable Solution** An approximate solution of our model problem is stable if it is less or equal than one and non-decreasing.

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- In order to get an stable solution we need

$$M > 5 \times 10^{2+k}.$$

Equidistant mesh (EQ-mesh)

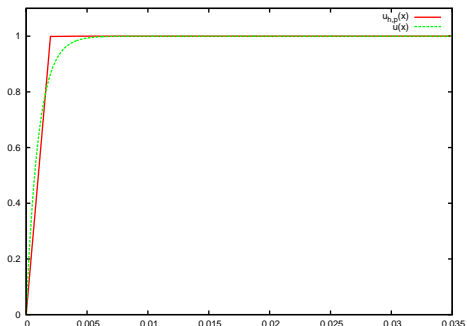


Figure: Stable approximate solution $u_{h,p}$, DOF=500, $\epsilon = 10^{-3}$, EQ-mesh, $Pe < 1$.

Equidistant mesh (EQ-mesh)

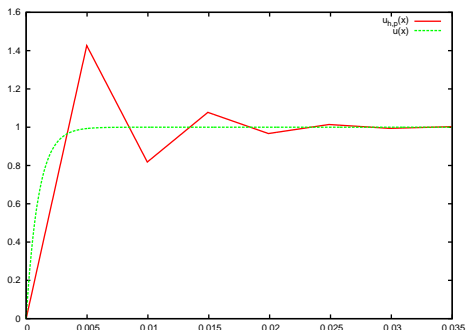


Figure: Detail of oscillations in the boundary layer, $\epsilon = 10^{-3}$, EQ-mesh, $Pe > 1$.

Adapted Equidistant Mesh (AEQ-mesh)

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- In AEQ-mesh setting with $h = 35\epsilon/M$, in $\Omega = (0, 35\epsilon)$, and one element in $(35\epsilon, 1)$, according to the Péclet Number we have:

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- For an stable solution the number of elements M satisfy $M > 35/2$.

Adapted Equidistant Mesh (AEQ-mesh)

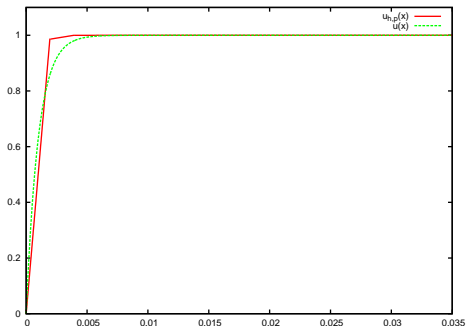


Figure: Stable approximate solution $u_{h,p}(x)$, $\text{DOF} = 18$, $\epsilon = 10^{-3}$, AEQ-mesh, $Pe < 1$.

Shishkin Mesh (S-mesh)

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- We define the Shishkin layer adapted mesh:

$$x_i = \begin{cases} \sigma\epsilon\varphi(t_i) & \text{with } t_i = i/M & \text{for } i = 0, 1, \dots, \frac{1}{2}M, \\ 35\epsilon - (35\epsilon - x_{N/2}) \cdot 2(M - i)/M & & \text{for } i = \frac{1}{2}M + 1, \dots, M. \end{cases}$$

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- Shishkin-mesh characterized by the mesh generation function

$$\varphi(t) = 2(\ln N)t \quad \text{with} \quad \lambda = \sigma\epsilon \ln N.$$

error.

Shishkin Mesh (S-mesh)

- $\sigma = 0.25$ yields the minimum value of H^1 -seminorm of the approximation

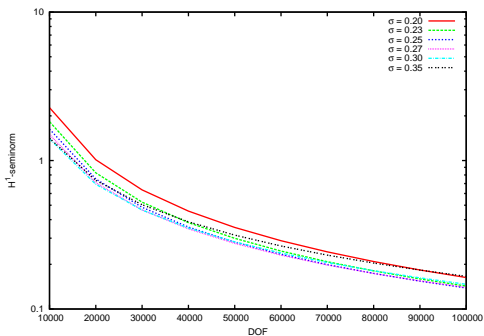


Figure: H^1 -seminorm of approximation error for various σ with $\epsilon = 10^{-8}$, S-mesh.

Bakhvalov Mesh (B-mesh)

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$$x_i = \begin{cases} \sigma\epsilon\varphi(t_i) \text{ with } t_i = i/M & \text{for } i = 0, 1, \dots, \frac{1}{2}M, \\ 35\epsilon - (35\epsilon - x_{M/2}) \cdot 2(M - i)/M & \text{for } i = \frac{1}{2}M + 1, \dots, M. \end{cases}$$

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- Consider a Bakhvalov-type mesh with mesh generating function:

$$\varphi(t) = -\ln[1 - 2(1 - \epsilon)t].$$

Bakhvalov Mesh (B-mesh)

- $\sigma = 1.5$ yields the smallest value of the H^1 -seminorm of the approximation error.

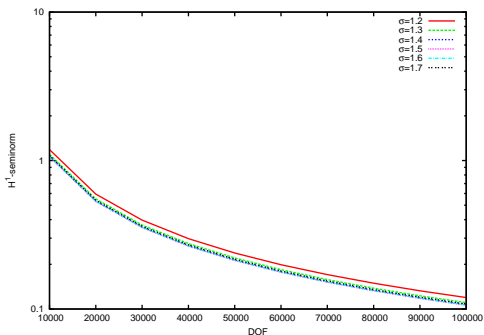


Figure: H^1 -seminorm of approximation error for σ with $\epsilon = 10^{-8}$, B-mesh.

Interpolation

- Given a point $x_i \in [0, 35\epsilon]$ and a distance h , we want to find a formula for the piecewise-linear interpolation error in the H^1 -seminorm in $[x_i, x_i + h]$.

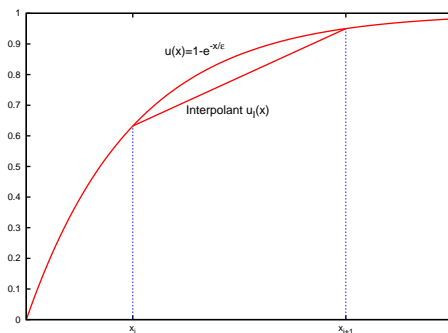


Figure: Piecewise-linear interpolant $u_I(x)$ in $[x_i, x_i + h]$.

Calculation of the Interpolation Error

- $u(x_i) = 1 - e^{-\frac{x_i}{\epsilon}}$, $u(x_i + h) = 1 - e^{-\frac{x_i+h}{\epsilon}}$. The slope m of the line interpolant $u_I(x)$ is

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- The H^1 -seminorm of the piecewise-linear interpolant u_I and u error, is given by:

$$|u - u_I|_{1,2}^2 = \int_{x_i}^{x_{i+1}} (u'(x) - m)^2 dx.$$

Calculation of the Interpolation Error

- We simplify

$$|u - u_I|_{1,2}^2 = e^{-\frac{2x_j}{\epsilon}} \left[\frac{1}{2\epsilon} \left(1 - e^{-\frac{2h}{\epsilon}} \right) - \frac{1}{h} \left(1 - e^{-\frac{h}{\epsilon}} \right)^2 \right].$$

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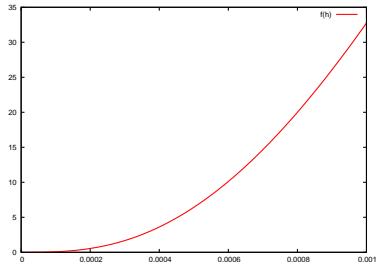
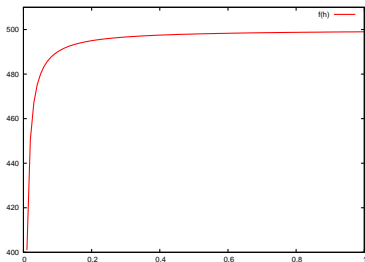
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$$|u - u_I|_{1,2}^2 = e^{-\frac{2x_j}{\epsilon}} f(h).$$

- We have an explicit formula for the H^1 -seminorm of the piecewise-linear interpolation error.

Calculation of Interpolation Error

- Graphs of the function of the interpolation error.



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- In order to obtain the length h_i of the next element, solve the nonlinear equation

$$f(h_i) = E_0 e^{\frac{2x_i}{\epsilon}}.$$

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- For the stopping criterion, it is important to note that

$$\lim_{h \rightarrow \infty} f(h) = 2\epsilon.$$

Generation of the Equidistributed Error Mesh

- We have solution if $E_0 e^{\frac{2x_i}{\epsilon}} \leq 2\epsilon$. Then

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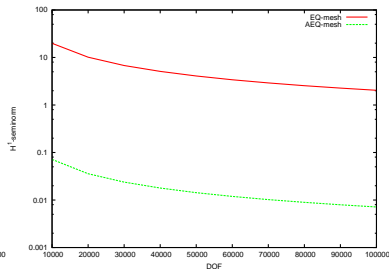
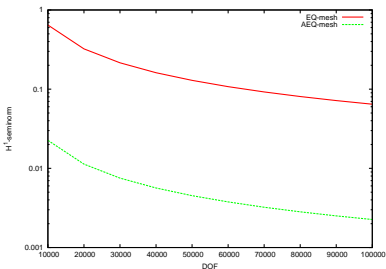
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- If $x_{M-1} < 35\epsilon$, construct one single linear element in $[x_{M-1}, 35\epsilon]$, and put a last element $[35\epsilon, 1]$. Otherwise put a last element $[x_{M-1}, 1]$.

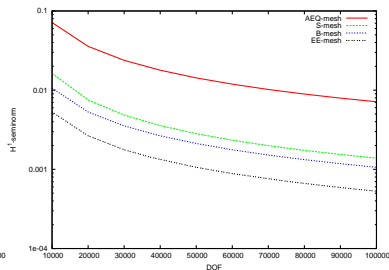
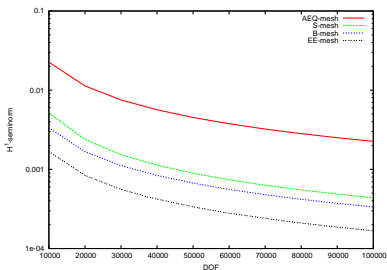
EQ-mesh vs. AEQ-mesh

- Graphs of H^1 -seminorm approximation error for $\epsilon = 10^{-3}$, $\epsilon = 10^{-4}$ for the FEM.



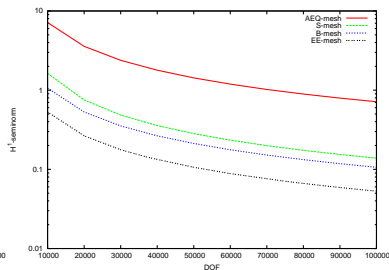
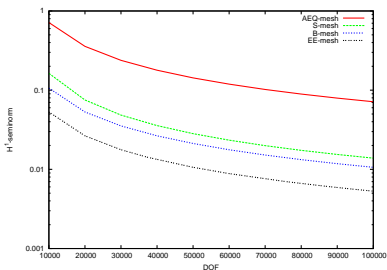
AEQ-mesh, S-mesh, B-mesh, EE-mesh

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AEQ-mesh, S-mesh, B-mesh, EE-mesh

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- For the new equidistributed error mesh, we have a better approximation than Shishkin and Bakhvalov meshes.

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- We cannot prove the convergence yet, we have done preliminary studies including large number of experiments and data fitting which resulted into the estimate of $\alpha = -1/2$.

Thank you!