On *hp*-FEM Based on Generalized Eigenfunctions

L. Dubcova, P. Solin

Department of Mathematical Sciences,
University of Texas at El Paso,

and Institute of Thermomechanics,
Academy of Sciences of the Czech Republic, Prague

FEMTEC, UTEP, El Paso, December 11, 2006

Outline

Higher-Order Shape Functions

Problem: Find $u_{hp} \in V_{hp}(\Omega)$ such that

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}(\Omega), \quad V_{hp} \subset V.$$

Express solution u_{hp} as a linear combination of basis functions:

$$u_{hp} = \sum_{j}^{N} c_j v_j.$$

Discrete problem: SY = F.
Choice of $\mathcal{B} = \{v_1, v_2, \ldots, v_N\}$ influences properties of S dramatically.

Example: Laplace Operator in $(-1, 1)$

Integrated Legendre polynomials

$$l_i(x) = \int_{-1}^{x} L_{i-1}(\xi)\,\mathrm{d}\xi, \qquad i = 2, 3, \ldots$$

H_0^1-product in $(-1, 1)$

$$(l_i, l_j)_{H_0^1(-1,1)} = \int_{-1}^{1} l_i'(\xi) l_j'(\xi)\,\mathrm{d}\xi = \delta_{ij}$$

L^2-product in $(-1, 1)$

$$(l_i, l_j)_{L^2(-1,1)} = \int_{-1}^{1} l_i(\xi) l_j(\xi)\,\mathrm{d}\xi \neq \delta_{ij}$$

Example: Laplace Operator in $(-1,1)^2$

Product shape functions in $K_q = (-1,1)^2$

$$\omega_{rs}(x_1, x_2) = l_r(x_1) l_s(x_2), \qquad 2 \le r, s \le p$$

H_0^1-product in $(-1,1)^2$ and L^2-product in $(-1,1)^2$

$$\begin{aligned}
(\omega_{ij}, \omega_{kl})_{H_0^1(K_q)} &= \int_{K_q} \nabla \omega_{ij}(x_1, x_2) \cdot \nabla \omega_{kl}(x_1, x_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \\
&= \delta_{ik} \int_{-1}^{1} l_j(x) l_l(x) \mathrm{d}x + \delta_{jl} \int_{-1}^{1} l_i(x) l_k(x) \mathrm{d}x \neq \delta_{ik}\delta_{jl} \\
(\omega_{ij}, \omega_{kl})_{L^2(K_q)} &= \int_{K_q} l_i l_j l_k l_l \, \mathrm{d}x_1 \mathrm{d}x_2 \neq \delta_{ik}\delta_{jl}
\end{aligned}$$

Generalized Eigenproblem for Laplace Operator

- Weak eigenproblem in $V = P_0^p(-1,1)$

$$\int_{-1}^{1} \psi'_m(x) v'(x)\, \mathrm{d}x = \lambda_m \int_{-1}^{1} \psi_m(x) v(x) \mathrm{d}x \qquad \text{for all } v \in V$$

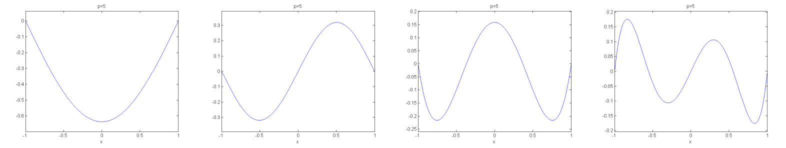
- Basis: $\mathcal{B}_p = \{g_1, g_2, \ldots, g_{p-1}\}$

$$\psi_k = \sum_{j=1}^{p-1} y_{jk} g_j$$

- Discrete problem: $SY = MY\Lambda$
- Analysis: Golub, Van Loan (e.g.)
- Solution: LAPACK, Matlab, ...

Gen. Eigenfunctions for the Laplacian in 1D

Example ($p = 5$)



H_0^1-product in $(-1, 1)$

$$(\psi_i, \psi_j)_{H_0^1(-1,1)} = \int_{-1}^{1} \psi_i'(\xi)\psi_j'(\xi)\,\mathrm{d}\xi = \delta_{ij}$$

L^2-product in $(-1, 1)$

$$(\psi_i, \psi_j)_{L^2(-1,1)} = \int_{-1}^{1} \psi_i(\xi)\psi_j(\xi)\,\mathrm{d}\xi = \frac{\delta_{ij}}{\lambda_i}$$

Gen. Eigenfunctions for the Laplacian in $\mathbb{R}^d$

Product shape functions in $K_d = (-1,1)^d$

$$\phi_{i_1,i_2,\ldots,i_d}(x_1,x_2,\ldots,x_d) = \left(\frac{\prod_{m=1}^d \lambda_{i_m}}{\sum_{m=1}^d \lambda_{i_m}}\right)^{1/2} \prod_{m=1}^d \psi_{i_m}(x_m) \qquad 1 \le i_m \le p-1$$

THEOREM: $\phi_{i_1,i_2,\ldots,i_d}$ are the gen. eigenfunctions of Δ in $Q_0^p(K_d)$.

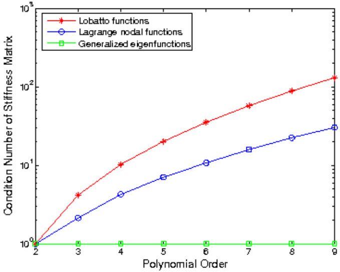
H_0^1-product in $(-1,1)^d$ and L^2-product in $(-1,1)^d$

$$(\phi_{i_1,i_2,\ldots,i_d}, \phi_{j_1,j_2,\ldots,j_d})_{H_0^1(K_d)} = \prod_{m=1}^d \delta_{i_m j_m}$$

$$(\phi_{i_1,i_2,\ldots,i_d}, \phi_{j_1,j_2,\ldots,j_d})_{L^2(K_d)} = \frac{\prod_{m=1}^d \delta_{i_m j_m}}{\sum_{m=1}^d \lambda_{i_m}}$$

Example: Product Element $K_q = (-1, 1)^2$

Comparison of condition numbers of stiffness matrix.



Time-Harmonic Maxwell's Equations

$$\boldsymbol{curl}(\mu^{-1}\mathrm{curl}\boldsymbol{E}) - \kappa^2\epsilon_r\boldsymbol{E} = \boldsymbol{F} \ \text{ in } \Omega,$$

- $\boldsymbol{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^T$
- $\mathrm{curl} = \partial E_2/\partial x_1 - \partial E_1/\partial x_2$
- $\boldsymbol{E} = \boldsymbol{E}(x) \in C^2$ electric field intensity
- $\boldsymbol{F} = \boldsymbol{F}(x) \in C^2$
- $\mu_r, \epsilon_r, \kappa$ permeability, permittivity and wave number

Boundary conditions:

- Perfect conductor boundary:

$$\boldsymbol{E} \cdot \tau = 0, \ on \ \Gamma_P$$

- Impedance boundary conditions:

$$(\mu^{-1}\mathrm{curl}\boldsymbol{E}) - i\kappa\lambda\boldsymbol{E} \cdot \tau = \boldsymbol{g} \cdot \tau, \quad \text{on} \quad \Gamma_I$$

Weak Formulation

$$
\begin{aligned}
a(\boldsymbol{E},\boldsymbol{F}) &= (\mu^{-1}\mathrm{curl}\boldsymbol{E},\mathrm{curl}\boldsymbol{F}) - \kappa^2(\epsilon_r\boldsymbol{E},\boldsymbol{F}) - i\kappa\langle\lambda\boldsymbol{E}\cdot\tau,\boldsymbol{F}\cdot\tau\rangle \\
F(\boldsymbol{F}) &= (\boldsymbol{f},\boldsymbol{F}) + \langle\boldsymbol{g}\cdot\tau,\boldsymbol{F}\cdot\tau\rangle
\end{aligned}
$$

Curl-curl product

$$\int_K \mathrm{curl}\phi_i(x,y)\cdot\mathrm{curl}\phi_j(x,y)\,\mathrm{d}x\mathrm{d}y = \delta_{ij}$$

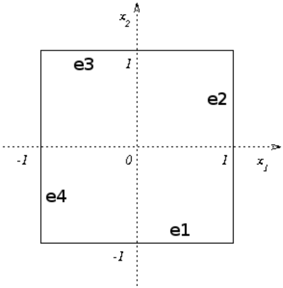
L^2-product

$$\int_K \phi_i(x,y)\cdot\phi_j(x,y)\,\mathrm{d}x\mathrm{d}y = D_{ij}$$

Shape Functions Based on Legendre Polynomials

Edge functions:

$$
\begin{aligned}
\phi_{k,q}^{e_1} &= L_k(\xi_1) l_0(\xi_2) \boldsymbol{e}_1, \quad 0 \le k \le p^{e_1} \\
\phi_{k,q}^{e_2} &= l_1(\xi_1) L_k(\xi_2) \boldsymbol{e}_2, \quad 0 \le k \le p^{e_2} \\
\phi_{k,q}^{e_3} &= L_k(\xi_1) l_1(\xi_2) \boldsymbol{e}_1, \quad 0 \le k \le p^{e_3} \\
\phi_{k,q}^{e_4} &= l_0(\xi_1) L_k(\xi_2) \boldsymbol{e}_2, \quad 0 \le k \le p^{e_4}
\end{aligned}
$$



where $\boldsymbol{e}_1, \boldsymbol{e}_2$ are canonical vectors.
Bubble functions:

$$
\begin{aligned}
\phi_{k_1,k_2,q}^{b,1} &= L_{k_1}(\xi_1) l_{k_2}(\xi_2) \boldsymbol{e}_1, \quad 0 \le k_1 \le p_1, 2 \le k_2 \le p_2 + 1 \\
\phi_{k_1,k_2,q}^{b,2} &= l_{k_1}(\xi_1) L_{k_2}(\xi_2) \boldsymbol{e}_2, \quad 2 \le k_1 \le p_1 + 1, 0 \le k_2 \le p_2
\end{aligned}
$$

Example: Curl-Curl Operator in $(-1,1)^2$

Curl-curl product in $(-1,1)^2$

$$
\begin{aligned}
(\phi_{i,j}^{b,1}, \phi_{k,l}^{b,2})_{H(\mathrm{curl})(K_q)} &= \int_{K_q} \mathrm{curl}\phi_{i,j}^{b,1}(x,y) \cdot \mathrm{curl}\phi_{k,l}^{b,2}(x,y)\,\mathrm{d}x\mathrm{d}y \\
&= -\int_{K_q} L_i(x) l_j'(y) l_k'(x) L_l(y)\mathrm{d}x\mathrm{d}y \neq 0
\end{aligned}
$$

L^2 product in $(-1,1)^2$

$$
\begin{aligned}
(\phi_{i,j}^{b,1}, \phi_{k,l}^{b,1})_{L^2(K_q)} &= \int_{K_q} L_i(x) l_j(y) L_k(x) l_l(y)\mathrm{d}x\mathrm{d}y \\
&= \delta_{ik} \int_{-1}^{1} l_j(y) l_l(y)\mathrm{d}y \neq \delta_{ik}\delta_{jl}
\end{aligned}
$$

Generalized Eigenproblem for Curl-Curl Operator

Finite edge element K is associated with the polynomial space Q_p. Solve the eigen-problem for Curl-Curl operator: find $\phi \in Q_p(K)$ such that

$$\int_K \mathrm{curl}\phi(x,y)\cdot\mathrm{curl}\psi(x,y)\,\mathrm{d}x\mathrm{d}y = \lambda \int_K \phi(x,y)\cdot\psi(x,y)\,\mathrm{d}x\mathrm{d}y \quad \forall\psi \in Q_p(K)$$

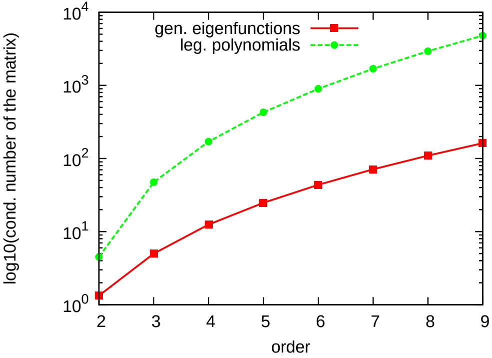
Properties - orthogonality in both products

$$\begin{aligned} \int_K \mathrm{curl}\phi_i(x,y)\cdot\mathrm{curl}\phi_j(x,y)\,\mathrm{d}x\mathrm{d}y &= 0, \quad i \neq j \\ \int_K \phi_i(x,y)\cdot\phi_j(x,y)\,\mathrm{d}x\mathrm{d}y &= 0 \quad i \neq j \end{aligned}$$

Example: Product Element $K_q = (-1, 1)^2$

On the product element we can express the eigen functions of Curl-Curl operator with the aid of the eigenfunctions for Laplace operator in 1D.
Comparison of condition numbers for the following matrix:

$$\int_{K_q} \mathrm{curl}\phi_i \, \mathrm{curl}\phi_j \, \mathrm{d}x\mathrm{d}y - \int_{K_q} \phi_i \phi_j \, \mathrm{d}x\mathrm{d}y, \qquad \phi_i, \phi_j \text{ - shape functions}$$



Example: L-Shape Domain - Diffraction problem

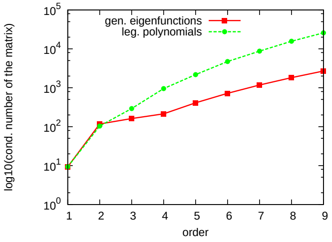
Time harmonic Maxwell's equations with exact solution

$$\begin{aligned} \textbf{curl}(\text{curl}\boldsymbol{E}) - \boldsymbol{E} &= \boldsymbol{F} \text{ in } \Omega, \\ \boldsymbol{E} \cdot \tau &= 0 \text{ on } \Gamma_P \\ \text{curl}\boldsymbol{E} - i\boldsymbol{E} \cdot \tau &= \boldsymbol{g} \cdot \tau, \text{ on } \Gamma_I \end{aligned}$$

Exact solution expressed by Bessel function

$$\boldsymbol{E} = \textbf{curl}(J_\alpha(r)\cos(\alpha\phi))$$

Example: L-Shape Domain - Diffraction problem



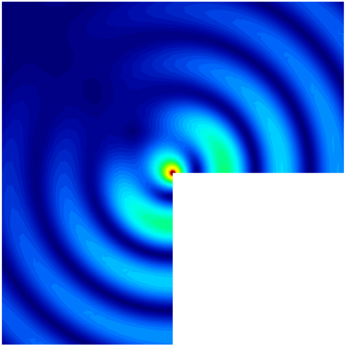


Figure: Comparison of condition numbers (left), magnitude of solution ***E***

De Rham Diagram

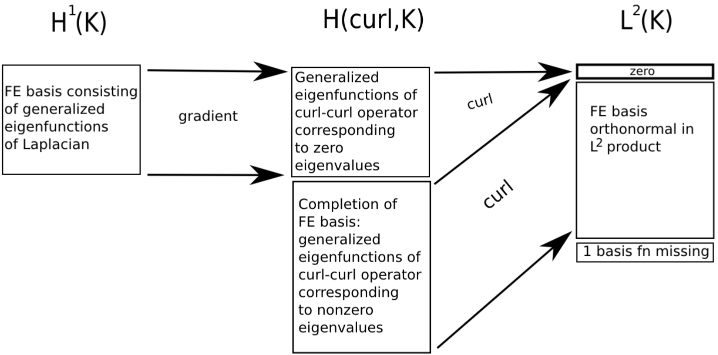
- Differential scheme relating spaces H^1, $\boldsymbol{H}(\mathrm{curl})$ and L^2:

$$H^1 \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{curl}) \xrightarrow{\nabla\times} L^2$$

- Finite elements should fit into this diagram: Important for stability of mixed FEM, convergence analysis, etc.

Finite Elements Based on Gen. Eigenfunctions

This class of finite elements naturally fits into the De Rham diagram. Let $K \subset \mathbb{R}^2$. Then



Conclusion and Outlook

FE Based on Generalized Eigenfunctions

- Maximum orthogonality:
  - H_0^1 and L^2 for second-order elliptic problems,
  - curl-curl and L^2 for Maxwell's equations.
- Excellent conditioning properties.
- Fit into the De Rham diagram.

Outlook

- Extension to 3D elements.
- *hp*-FEM for coupled problems.

Thank you for your attention.