Continuous *hp* Elements Based on Generalized Eigenfunctions

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Outline

Higher-Order Shape Functions

Integrated Legendre Polynomials

$$l_i(x) = \int_{-1}^{x} L_{i-1}(\xi)\, \mathrm{d}\xi, \qquad i = 2, 3, \ldots$$

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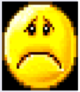

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  - Integrated Legendre polynomials
  - Arbitrary basis + Gram–Schmidt
  - Nonuniqueness = room for additional conditions

Another Set of Orthonormal Polynomials

- Weak eigenproblem in $V = P_0^p(-1,1)$

$$\int_{-1}^{1} \psi_m'(x)v'(x)\,\mathrm{d}x = \lambda_m \int_{-1}^{1} \psi_m(x)v(x)\mathrm{d}x \qquad \text{for all } v \in V$$

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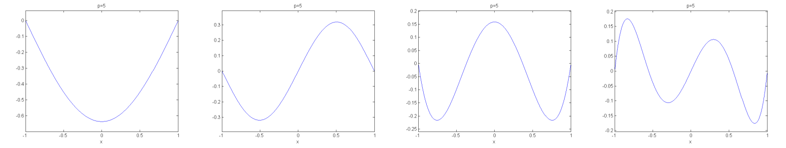
- Solution: LAPACK, Matlab, ...

Properties – 1D

Example ($p = 5$)



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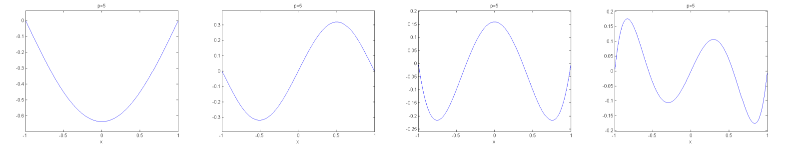

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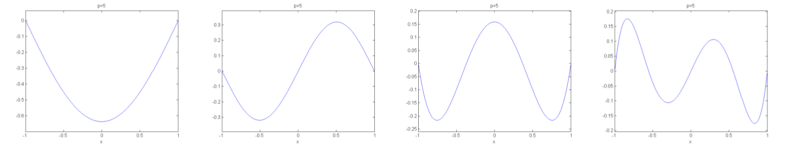

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Conditioning Comparison

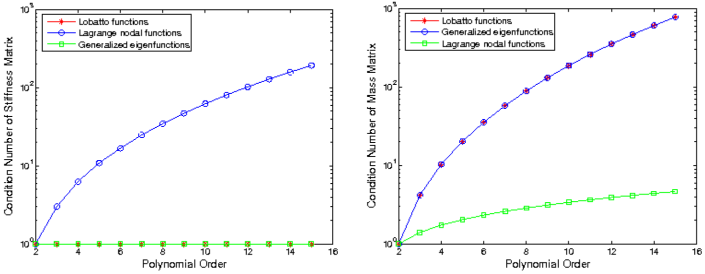

Figure: Conditioning of the stiffness and mass matrices, $p = 2, 3, \ldots, 15$.

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Then for any inner product $b : W \times W \to \mathbb{R}$, matrices $M_\gamma = \{b(\gamma_i, \gamma_j)\}_{i,j=1}^n$ and $M_\omega = \{b(\omega_i, \omega_j)\}_{i,j=1}^n$ have the same set of eigenvalues.

PROOF: See P. Šolín, T. Vejchodský: Continuous *hp* Elements Based on Generalized Eigenfunctions of the Laplacian, Research Report No. 2006-08, Department of Math. Sciences, University of Texas at El Paso, 2006.

Properties – Product Elements in $\mathbb{R}^d$

Product Shape Functions in $K_d = (-1,1)^d$

$$\phi_{i_1,i_2,\ldots,i_d}(x_1, x_2, \ldots, x_d) = \left(\frac{\prod_{m=1}^{d} \lambda_{i_m}}{\sum_{m=1}^{d} \lambda_{i_m}} \right)^{1/2} \prod_{m=1}^{d} \psi_{i_m}(x_m) \qquad 1 \le i_m \le p-1$$

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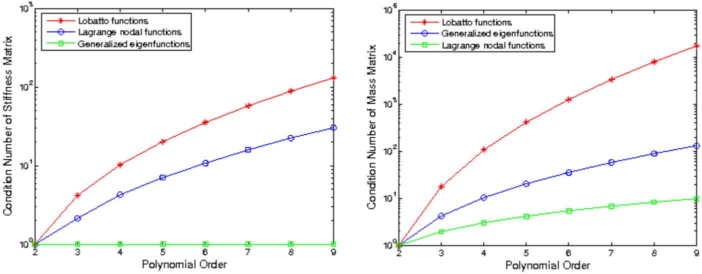

Figure: Conditioning of the stiffness and mass matrices, $p = 2, 3, \ldots, 9$.

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- Matrix form: $MY = SY\Lambda$

Example: Triangle $K_t = ([-1,-1],[1,-1],[-1,1])$

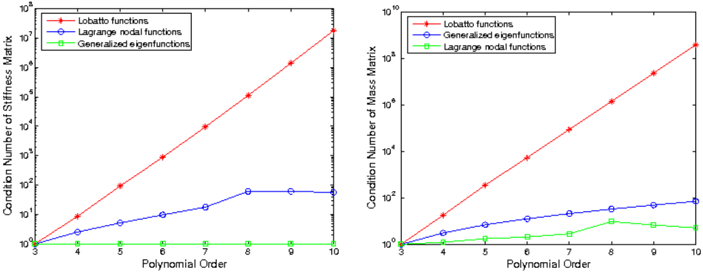

Figure: Conditioning of the stiffness and mass matrices, $p = 3, 4, \ldots, 10$.

Example: L-Shape Domain Problem

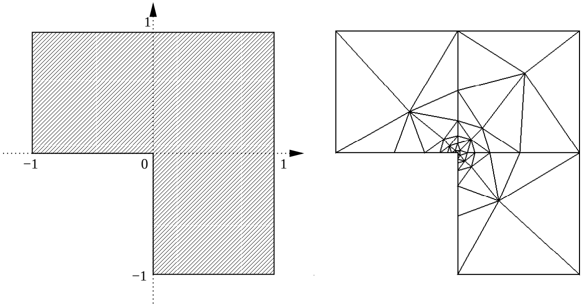

Figure: The L-shape domain and its partition.

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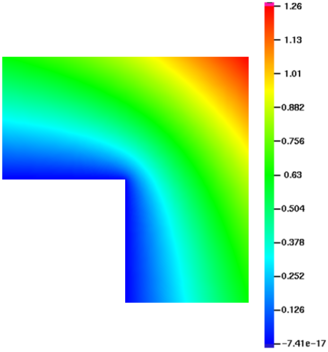
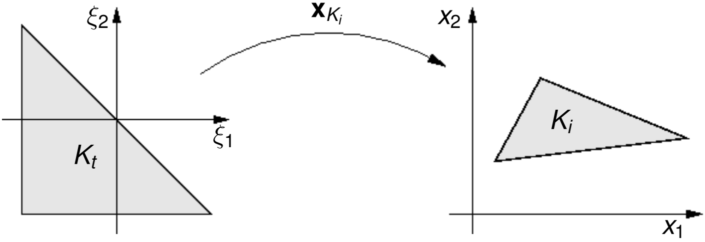


Figure: Exact solution *u* (left) and the norm of its gradient (right).

Role of Reference Maps



Several ways to map the central vertex of K_t:

- vertex w. *lowest index ("random")* in K_i
- vertex w. *minimum angle* in K_i
- vertex w. *medium angle* of K_i
- vertex w. *maximum angle* of K_i

Lobatto Shape Functions

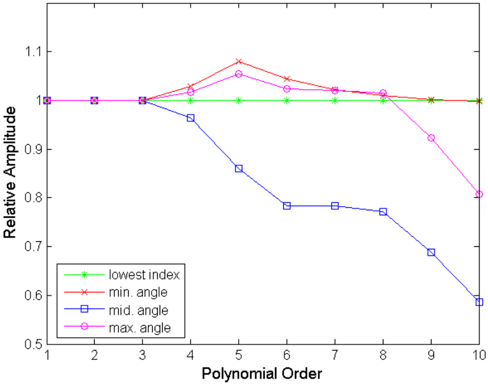
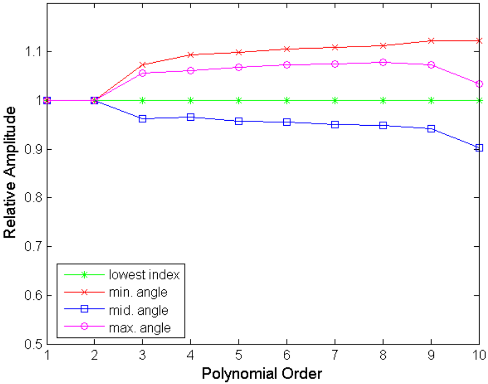


Figure: Condition number of stiffness matrix, $p = 1, 2, \ldots, 10$.

Jacobi Shape Functions



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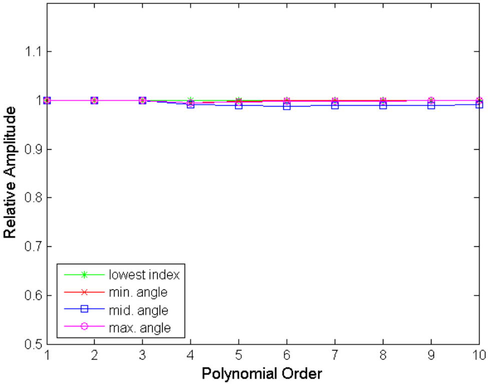
Generalized Eigenfunctions



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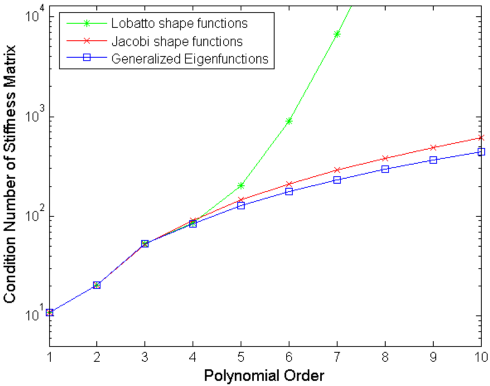

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Conclusion and Outlook

Generalized Eigenfunctions of Δ

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Outlook

- Application in analysis (DMP, ...)
- Natural also for edge elements $\rightarrow$ talk by T. Vejchodský
- Stokes, linear convection-diffusion, Navier-Stokes, ...
- Monolithic *hp*-FEM for coupled problems