

On the hp -FEM for Time-Harmonic Maxwell's Equations

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1 Higher-Order Basis Functions Based on Generalized Eigenfunctions

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- Standard Basis
- Generalized Eigenfunctions
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- How to Handle Hanging Nodes
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3 Curvilinear Elements

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- Numerical Quadrature and Curvilinear Elements
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Higher-Order Shape Functions

Problem: Find $u_{hp} \in V_{hp}(\Omega)$ such that

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}(\Omega), \quad V_{hp} \subset V.$$

Express solution u_{hp} as a linear combination of basis functions:

$$u_{hp} = \sum_j^N c_j v_j.$$

Discrete problem: $SY = F$.

Choice of $\mathcal{B} = \{v_1, v_2, \dots, v_N\}$ influences properties of S dramatically.

Curl-Curl Product, Space $H(\text{curl})$

Time-Harmonic Maxwell's Equations:

$$\mathbf{curl}(\mu^{-1} \mathbf{curl} \mathbf{E}) - \kappa^2 \epsilon_r \mathbf{E} = \mathbf{F} \text{ in } \Omega,$$

- $\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^T$
- $\mathbf{curl} = \partial E_2/\partial x_1 - \partial E_1/\partial x_2$
- $\mathbf{E} = \mathbf{E}(x) \in C^2$ electric field intensity
- $\mathbf{F} = \mathbf{F}(x) \in C^2$
- $\mu_r, \epsilon_r, \kappa$ permeability, permittivity and wave number

Boundary conditions:

- Perfect conductor boundary:

$$\mathbf{E} \cdot \boldsymbol{\tau} = 0, \text{ on } \Gamma_P$$

- Impedance boundary conditions:

$$(\mu^{-1} \mathbf{curl} \mathbf{E}) - i\kappa\lambda \mathbf{E} \cdot \boldsymbol{\tau} = \mathbf{g} \cdot \boldsymbol{\tau}, \text{ on } \Gamma_I$$

Weak Formulation

Find $\mathbf{E} \in H(\text{curl}, \Omega)$ such that

$$a(\mathbf{E}, \mathbf{F}) = F(\mathbf{F}), \text{ for all } \mathbf{F} \in H(\text{curl}, \Omega)$$

where

$$a(\mathbf{E}, \mathbf{F}) = (\mu^{-1} \text{curl} \mathbf{E}, \text{curl} \mathbf{F}) - \kappa^2 (\epsilon_r \mathbf{E}, \mathbf{F}) - i\kappa \langle \lambda \mathbf{E} \cdot \boldsymbol{\tau}, \mathbf{F} \cdot \boldsymbol{\tau} \rangle$$

$$F(\mathbf{F}) = (\mathbf{f}, \mathbf{F}) + \langle \mathbf{g} \cdot \boldsymbol{\tau}, \mathbf{F} \cdot \boldsymbol{\tau} \rangle$$

We would like to find functions which are simultaneously orthogonal both in the curl-curl and L^2 products which would make them an excellent basis for the hp -FEM.

Curl-curl product

$$\int_K \text{curl} \phi_i(x, y) \cdot \text{curl} \phi_j(x, y) \, dx dy = \delta_{ij}$$

L^2 -product

$$\int_K \phi_i(x, y) \cdot \phi_j(x, y) \, dx dy = D_{ij}$$

Standard Basis: Shape Functions Based on Legendre Polynomials

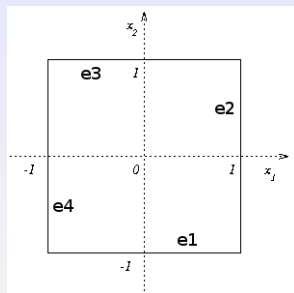
Bubble functions:

$$\phi_{k_1, k_2, q}^{b,1} = L_{k_1}(\xi_1) l_{k_2}(\xi_2) \mathbf{e}_1,$$

$$0 \leq k_1 \leq p_1, 2 \leq k_2 \leq p_2 + 1$$

$$\phi_{k_1, k_2, q}^{b,2} = l_{k_1}(\xi_1) L_{k_2}(\xi_2) \mathbf{e}_2,$$

$$2 \leq k_1 \leq p_1 + 1, 0 \leq k_2 \leq p_2$$



where \mathbf{e}_1 and \mathbf{e}_2 are canonical vectors.

Curl-curl product and L^2 -product in $(-1, 1)^2$

$$\int_{K_q} \text{curl} \phi_{i,j}^b(x, y) \cdot \text{curl} \phi_{k,l}^b(x, y) dx dy \neq \delta_{ik} \delta_{jl}$$

$$\int_{K_q} \phi_{i,j}^b(x, y) \phi_{k,l}^b(x, y) dx dy \neq \delta_{ik} \delta_{jl}$$

Improvement: Generalized Eigenproblem for Curl-Curl Operator

Solve the eigen-problem for Curl-Curl operator: find $\phi \in \mathbf{Q}_p(K)$ such that

$$\int_K \operatorname{curl} \phi(x, y) \cdot \operatorname{curl} \psi(x, y) \, dx dy = \lambda \int_K \phi(x, y) \cdot \psi(x, y) \, dx dy \quad \forall \psi \in \mathbf{Q}_p(K)$$

Basis: $\mathcal{B}_p = \{g_1, g_2, \dots, g_{p-1}\}$

$$\phi_k = \sum_{j=1}^{p-1} y_{jk} g_j$$

- Discrete problem: $SY = MY\Lambda$
- Solution: LAPACK, Matlab, ...

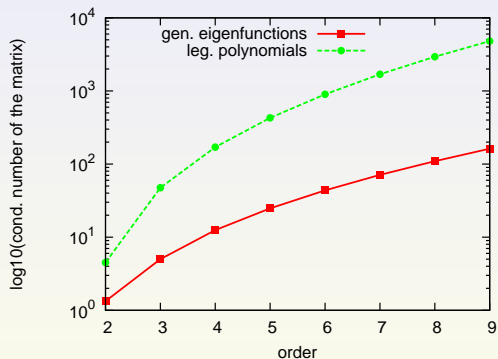
Properties - orthogonality in both products

$$\begin{aligned} \int_K \operatorname{curl} \phi_i(x, y) \cdot \operatorname{curl} \phi_j(x, y) \, dx dy &= 0, \quad i \neq j \\ \int_K \phi_i(x, y) \cdot \phi_j(x, y) \, dx dy &= 0 \quad i \neq j \end{aligned}$$

Condition Numbers - Product Element $K_q = (-1, 1)^2$

Comparison of condition numbers for the following matrix:

$$\int_{K_q} \text{curl} \phi_i \text{curl} \phi_j \, dx dy - \int_{K_q} \phi_i \phi_j \, dx dy, \quad \phi_i, \phi_j - \text{bubble functions}$$



Example: L-Shape Domain - Diffraction problem

Refined mesh \rightarrow the matrix is affected by

- the edge shape functions - both sets have same edge functions, they differ only in bubble functions
- by the shapes of the elements

But conditioning properties of the generalized eigenfunctions are still better than other used sets of shape functions.

Time-harmonic Maxwell's equations with exact solution

$$\begin{aligned}\mathbf{curl}(\mathbf{curl}\mathbf{E}) - \mathbf{E} &= \mathbf{F} \text{ in } \Omega, \\ \mathbf{E} \cdot \boldsymbol{\tau} &= 0 \text{ on } \Gamma_P \\ \mathbf{curl}\mathbf{E} - i\mathbf{E} \cdot \boldsymbol{\tau} &= \mathbf{g} \cdot \boldsymbol{\tau}, \text{ on } \Gamma_I\end{aligned}$$

Exact solution expressed by Bessel function

$$\mathbf{E} = \mathbf{curl}(J_\alpha(r) \cos(\alpha\phi))$$

Example: L-Shape Domain - Diffraction problem

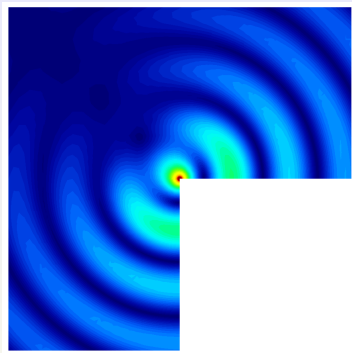
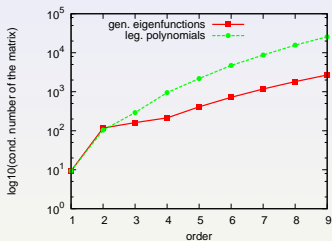


Figure: Comparison of condition numbers (left), magnitude of solution E

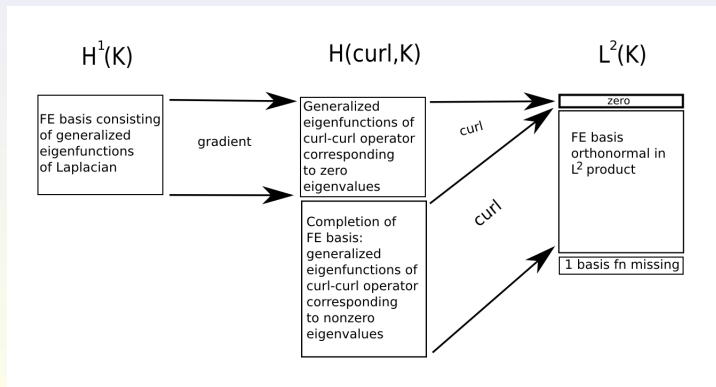
De Rham Diagram and Finite Elements Based on Gen. Eigenfunctions

- Differential scheme relating spaces H^1 , $\mathbf{H}(\text{curl})$ and L^2 :

$$H^1 \xrightarrow{\nabla} \mathbf{H}(\text{curl}) \xrightarrow{\nabla \times} L^2$$

This class of finite elements naturally fits into the De Rham diagram.

Let $K \subset \mathbb{R}^2$. Then



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Conformity Requirements of the Space $\mathbf{H}(\text{curl})$

Conformity requirements of the vector valued space $\mathbf{H}(\text{curl}, \Omega_{hp})$:

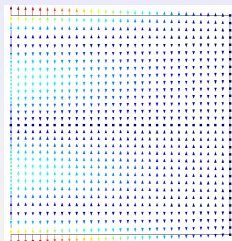
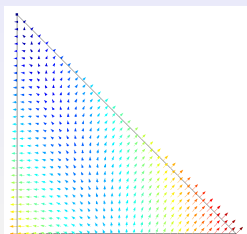
suppose $\mathbf{E} \in \mathbf{H}(\text{curl}, \Omega_{hp})$

- $\mathbf{E}|_K \in [H^1(K)]^2$ for each element K
- For each edge $e = K_1 \cap K_2$ the traces of the tangential components are same:

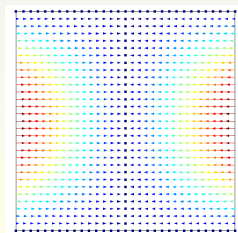
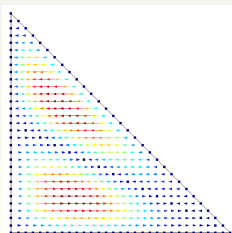
$$\mathbf{E}|_{K_1} \times \mathbf{t}_e = \mathbf{E}|_{K_2} \times \mathbf{t}_e$$

Shape Functions

- no vertex basis functions
- $p + 1$ edge functions for polynomial degree p (associated with edge)



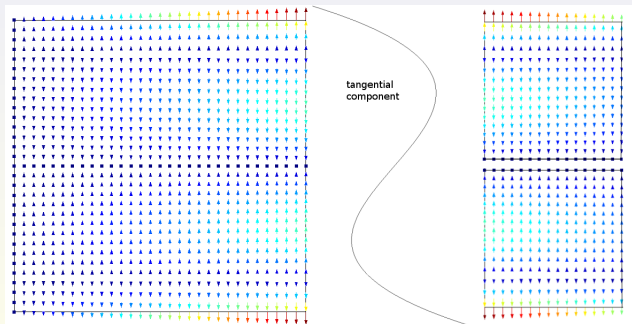
- bubble functions (starting at pol. degree 2)



Hanging nodes

- edges:

- ▶ edge shape functions have to satisfy conformity requirements = tangential components have to be the same at the interface of two elements
- ▶ for edge element with pol. degree p – tangential components of all edge shape functions are of degree $0, 1, \dots, p$
- ▶ tangential component of each part of the constraining edge function has to be expressed by a linear combination of tangential components of edge functions, which are attached to this part of the edge \Rightarrow tangential components will be the same
- ▶ all these linear combinations of edge shape functions together with the constraining function make up one basis function, which fulfill conformity requirements of the space $\mathbf{H}(\text{curl})$.



- bubbles:

- ▶ no problem with bubble functions - the support is only one element, no conformity requirements

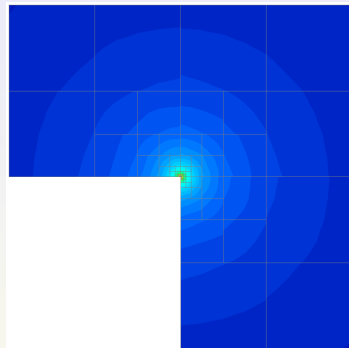
Test Example: L-Shape Domain

Time-harmonic Maxwell's equations
with exact solution

$$\begin{aligned}\mathbf{curl}(\mathbf{curl}\mathbf{E}) - \mathbf{E} &= \mathbf{F} \text{ in } \Omega, \\ \mathbf{E} \cdot \boldsymbol{\tau} &= 0 \text{ on } \Gamma_P \\ \mathbf{curl}\mathbf{E} - i\mathbf{E} \cdot \boldsymbol{\tau} &= \mathbf{g} \cdot \boldsymbol{\tau}, \text{ on } \Gamma_I\end{aligned}$$

Exact solution

$$\mathbf{E} = 2/3r^{-1/3} \begin{bmatrix} \cos(\frac{\pi}{6} + \frac{\theta}{3}) \\ \sin(\frac{\pi}{6} + \frac{\theta}{3}) \end{bmatrix}$$



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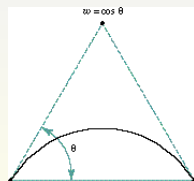
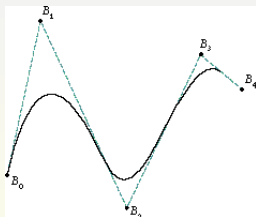
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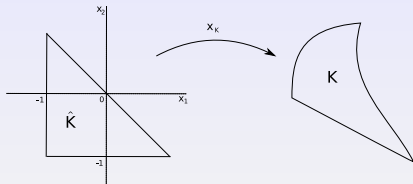
NURBS Curves

- can accurately represent standard geometric objects like lines, circles, ellipses, parabolas and hyperbolas,
- are generalizations of non-rational B-splines and non-rational and rational Bezier curves,
- can be evaluated reasonably fast by numerically stable and accurate algorithms,
- defined by three things:
 - ▶ degree k (usually 1,2,3,5),
 - ▶ control points P_i , weights w_i , $i = 0, \dots, n - 1$,
 - ▶ knot vector $\{t_0, \dots, t_{m-1}\}$.

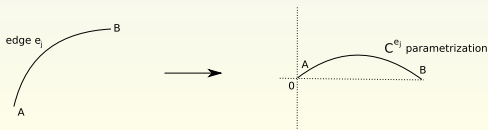


Reference Mapping

Reference mapping for curvilinear elements: $X_K = X_K^v + X_K^e$



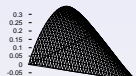
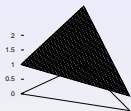
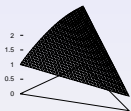
- affine part is the same as for straight elements: $X_K^v = \sum_{i=1}^3 x_i \varphi^i$ (physical mesh vertices and vertex functions)
- higher order part: $X_K^e = \sum_{j=1}^3 C^{ej} \lambda_A \lambda_B$, where C^{ej} is the parametrization of the curved edge (straight part was subtracted) and λ_A, λ_B are vertex functions corresponding to vertices A and B .



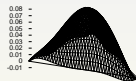
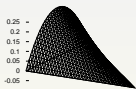
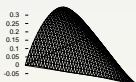
Isoparametric Approximation

isoparametric approximation = polynomial map defined in terms of master element shape functions

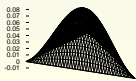
- Vertex interpolant $X_{h,p}^v$ - linear combination of the vertex shape functions



- Edge interpolant $X_{h,p}^e$ - by projecting the residual $X - X_{h,p}^v$ on the polynomial space generated by the edge functions



- Bubble interpolant $X_{h,p}^b$ - by projecting the residual $X - X_{h,p}^v - X_{h,p}^e$ on the polynomial space generated by the bubble functions



Example: transformation to the reference domain

$$\int_K \nabla \varphi_i \nabla \varphi_j \, dx = \int_{\hat{K}} J_{x_K} \left(\frac{dx_K}{d\xi} \right)^{-1} \nabla \hat{\varphi}_i \left(\frac{dx_K}{d\xi} \right)^{-1} \nabla \hat{\varphi}_j \, d\xi$$

To integrate this expression

- determine polynomial degree = polynomial degree of $\hat{\varphi}_i$ + polynomial degree of $\hat{\varphi}_j$ + additional degree necessary for the inverse mapping and Jacobian
- use accurate quadrature rule

- for linear triangles reference mapping is constant – OK (no additional degree)
- for quads and arbitrary curvilinear elements the inverse reference mapping is even nonpolynomial (for example: quads with small angles, curved elements with angle almost $180^\circ, \dots$)
 - ▶ try better and better quadrature rule until the integral converges
 - ▶ if we don't have enough quadrature rules \Rightarrow warn the user
 - ▶ or use adaptive quadrature in critical elements - refine the element domain until the integration error is small

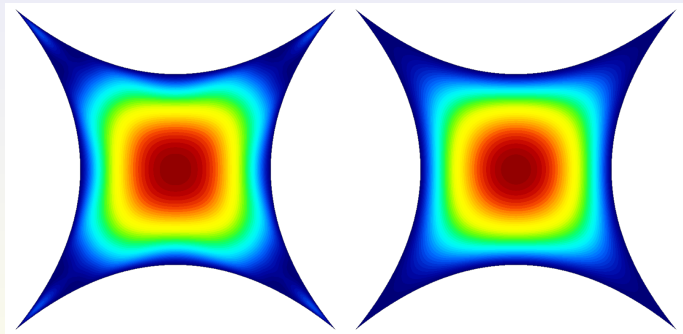
Influence of numerical quadrature I

$-\Delta u = f$, $f(x) = 1$, $u = 0$ on $\partial\Omega$, $p = 9$:

Gauss quadrature, order:

(a) $9 + 9 + 1$,

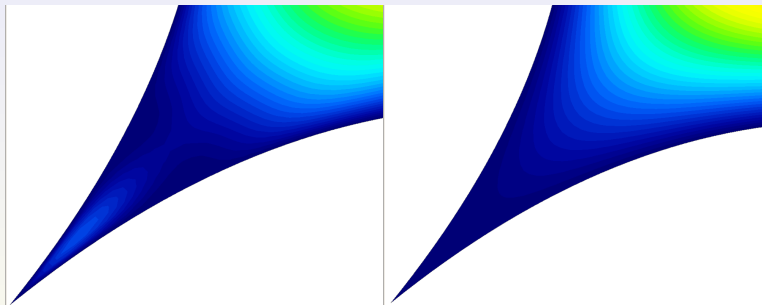
(b) $9 + 9 + 6$.



⇒ Need for adaptive quadrature!

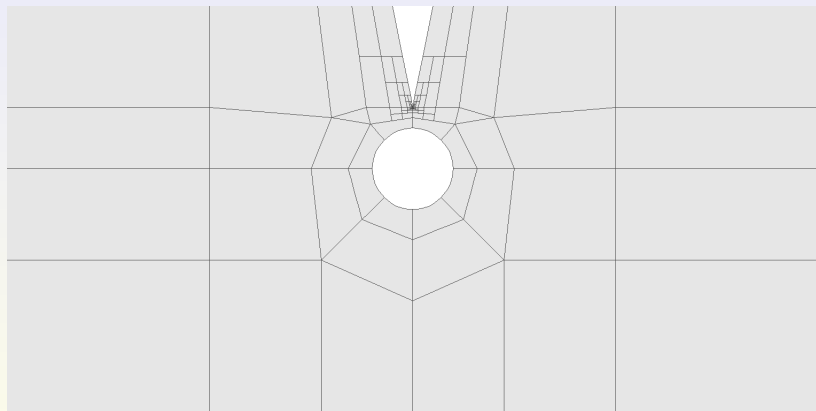
Influence of numerical quadrature II

Detail of a corner:



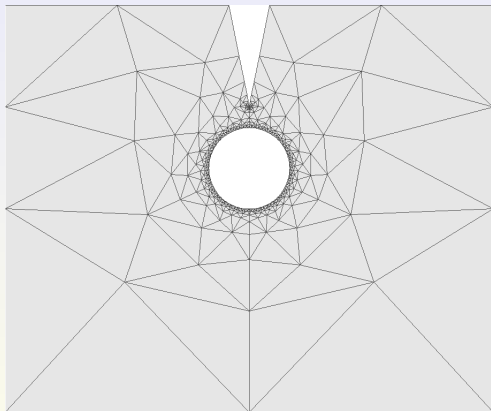
Sphere-Cone Problem - H^1 problem

A metallic sphere carries an electric potential $\varphi_S = 100kV$. To describe the surface of the sphere we need only 8 curvilinear elements. The mesh is refined near the peak of the cone (singularity of the gradient of the solution)



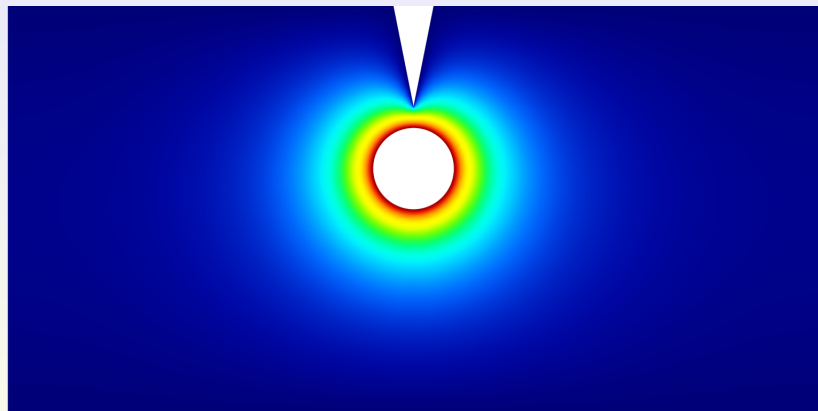
Sphere-Cone Problem

To describe the surface of the sphere we need hundreds of straight elements (only because of the curved boundary - not necessary for the solution).



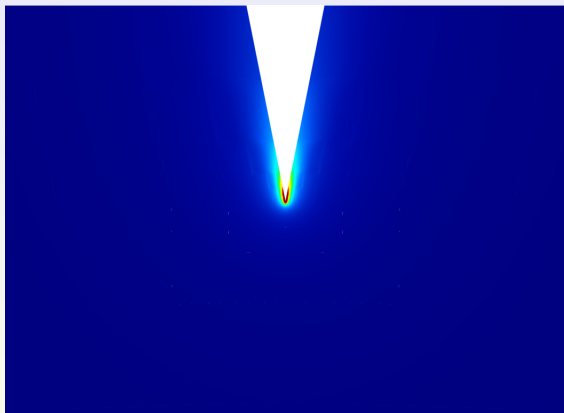
Sphere-Cone Problem

Solution



Sphere-Cone Problem

Gradient of the solution



Comparison

	Linear elements	<i>hp</i> elements	curved <i>hp</i> elements
DOF	488542	3317	2867
Error	0.5858%	0.2804%	0.2758%
Iterations	859	44	35
CPU time	30 min	10.53 sec	7.33 sec

Thank you for your attention