On the *hp*-FEM for Time-Harmonic Maxwell's Equations

Lenka Dubcova

February 18, 2007

Outline

Finite Element Method, *hp* - version

Weak formulation of PDE: Find $u \in V(\Omega), \Omega \subset \mathbb{R}^2$ such that

$$a(u, v) = F(v) \quad \forall v \in V(\Omega),$$

- approximation of the boundary of Ω: polygonal domain Ω_h
- partition Ω_h to finite number of simple elements K (triangles, quads)
- approximate the space V by a finite dimensional space V_{hp} (polynomial space, $\dim = N$, different polynomial degrees on elements)

Discrete problem: Find $u_{hp} \in V_{hp}(\Omega)$ such that

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}(\Omega).$$

- define a basis of the space V_{hp}: basis functions $v_i, i = 1, \ldots N$
- express solution u_{hp} as a linear combination of the basis functions $u_{hp} = \sum_j^N y_j v_j$
- discrete problem has to hold for all basis functions v_i

$$\sum_j^N a(v_j, v_i) y_j = F(v_i)$$

- solve linear system SY = F

Laplace Operator, Space H^1

Discrete problem: Find $u_{hp} \in V_{hp}(\Omega)$ such that

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}(\Omega).$$

where V_{hp} is a finite-dimensional approximation of the space H^1 and $a(u, v)$ is a bilinear form

$$a(u, v) = (\nabla u, \nabla v) + (u, v)$$

We would like to find functions which are simultaneously orthogonal both in the H_0^1 and L^2 products.

H_0^1 product

$$\int_K \nabla v_i(x, y) \cdot \nabla v_j(x, y)\, \mathrm{d}x\mathrm{d}y = \delta_{ij}$$

L^2-product

$$\int_K v_i(x, y) \cdot v_j(x, y)\, \mathrm{d}x\mathrm{d}y = D_{ij}$$

Higher-Order Basis Functions - Standard Basis

Choice of $\mathcal{B} = \{v_1, v_2, \ldots, v_N\}$ influences the properties of S dramatically.

Example: Laplace Operator in $(-1, 1)$

Integrated Legendre polynomials

$$l_i(x) = \int_{-1}^{x} L_{i-1}(\xi)\,\mathrm{d}\xi, \qquad i = 2, 3, \ldots$$

H_0^1-product in $(-1, 1)$

$$(l_i, l_j)_{H_0^1(-1,1)} = \int_{-1}^{1} l_i'(\xi) l_j'(\xi)\,\mathrm{d}\xi = \delta_{ij}$$

L^2-product in $(-1, 1)$

$$(l_i, l_j)_{L^2(-1,1)} = \int_{-1}^{1} l_i(\xi) l_j(\xi)\,\mathrm{d}\xi \neq \delta_{ij}$$

Situation in the square $(-1, 1)^2$ is even worse (basis functions defined as the product of these).

Better Basis - Generalized Eigenfunctions

Generalized eigenproblem for the Laplace operator:

- Weak eigenproblem in $V = P_0^p(-1,1)$

$$\int_{-1}^{1} \psi_m'(x)v'(x)\,\mathrm{d}x = \lambda_m \int_{-1}^{1} \psi_m(x)v(x)\mathrm{d}x \qquad \text{for all } v \in V$$

- Basis: $\mathcal{B}_p = \{g_1, g_2, \dots, g_{p-1}\}$

$$\psi_k = \sum_{j=1}^{p-1} y_{jk} g_j$$

- Discrete problem: $SY = MY\Lambda$
- Analysis: Golub, Van Loan (e.g.)
- Solution: LAPACK, Matlab, ...

Properties:

$$(\psi_i, \psi_j)_{H_0^1(-1,1)} = \int_{-1}^{1} \psi_i'(\xi)\psi_j'(\xi)\,\mathrm{d}\xi = \delta_{ij}$$

$$(\psi_i, \psi_j)_{L^2(-1,1)} = \int_{-1}^{1} \psi_i(\xi)\psi_j(\xi)\,\mathrm{d}\xi = \frac{\delta_{ij}}{\lambda_i}$$

Curl-Curl Product, Space $H(\text{curl})$

Time-Harmonic Maxwell's Equations:

$$\mathbf{curl}(\mu^{-1}\text{curl}\boldsymbol{E}) - \kappa^2\epsilon_r\boldsymbol{E} = \boldsymbol{F} \text{ in } \Omega,$$

- $\mathbf{curl} = (\partial/\partial x_2, -\partial/\partial x_1)^T$
- $\text{curl} = \partial E_2/\partial x_1 - \partial E_1/\partial x_2$
- $\boldsymbol{E} = \boldsymbol{E}(x) \in C^2$ electric field intensity
- $\boldsymbol{F} = \boldsymbol{F}(x) \in C^2$
- $\mu_r, \epsilon_r, \kappa$ permeability, permittivity and wave number

Boundary conditions:

- Perfect conductor boundary:

$$\boldsymbol{E} \cdot \tau = 0, \quad \text{on } \Gamma_P$$

- Impedance boundary conditions:

$$(\mu^{-1}\text{curl}\boldsymbol{E}) - i\kappa\lambda\boldsymbol{E} \cdot \tau = \boldsymbol{g} \cdot \tau, \quad \text{on } \Gamma_I$$

Weak Formulation

Find $\boldsymbol{E} \in H(\text{curl}, \Omega)$ such that

$$a(\boldsymbol{E}, \boldsymbol{F}) = F(\boldsymbol{F}), \text{ for all } \boldsymbol{F} \in H(\text{curl}, \Omega)$$

where

$$\begin{aligned} a(\boldsymbol{E}, \boldsymbol{F}) &= (\mu^{-1}\text{curl}\boldsymbol{E}, \text{curl}\boldsymbol{F}) - \kappa^2(\epsilon_r \boldsymbol{E}, \boldsymbol{F}) - i\kappa\langle\lambda \boldsymbol{E}\cdot\tau, \boldsymbol{F}\cdot\tau\rangle \\ F(\boldsymbol{F}) &= (\boldsymbol{f}, \boldsymbol{F}) + \langle \boldsymbol{g}\cdot\tau, \boldsymbol{F}\cdot\tau\rangle \end{aligned}$$

We would like to find functions which are simultaneously orthogonal both in the curl-curl and L^2 products which would make them an excellent basis for the *hp*-FEM.

Curl-curl product

$$\int_K \text{curl}\phi_i(x, y) \cdot \text{curl}\phi_j(x, y)\, \text{d}x\text{d}y = \delta_{ij}$$

L^2-product

$$\int_K \phi_i(x, y) \cdot \phi_j(x, y)\, \text{d}x\text{d}y = D_{ij}$$

Conformity Requirements of the Space $\boldsymbol{H}$(curl)

Conformity requirements of the space $\boldsymbol{H}(\text{curl}, \Omega_{hp})$:

suppose $\boldsymbol{E} \in \boldsymbol{H}(\text{curl}, \Omega_{hp})$

- $\boldsymbol{E}|_K \in \left[H^1(K)\right]^2$ for each element $K \Rightarrow$ continuity on elements
- For each edge $e = K_1 \cap K_2$ the traces of the tangential components are same:

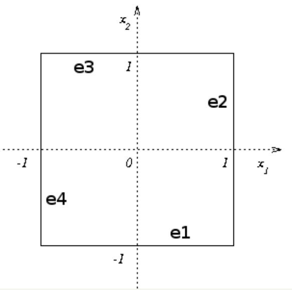
$$\boldsymbol{E}|_{K_1} \times \boldsymbol{t}_e = \boldsymbol{E}|_{K_2} \times \boldsymbol{t}_e$$

$\Rightarrow$ continuity of tangential component across the edges

Standard Basis: Shape Functions Based on Legendre Polynomials

Edge functions:

$$
\begin{aligned}
\phi^{e_1}_{k,q} &= L_k(\xi_1) l_0(\xi_2)\mathbf{e}_1, \quad 0 \le k \le p^{e_1} \\
\phi^{e_2}_{k,q} &= l_1(\xi_1) L_k(\xi_2)\mathbf{e}_2, \quad 0 \le k \le p^{e_2} \\
\phi^{e_3}_{k,q} &= L_k(\xi_1) l_1(\xi_2)\mathbf{e}_1, \quad 0 \le k \le p^{e_3} \\
\phi^{e_4}_{k,q} &= l_0(\xi_1) L_k(\xi_2)\mathbf{e}_2, \quad 0 \le k \le p^{e_4}
\end{aligned}
$$



where $\mathbf{e}_1, \mathbf{e}_2$ are canonical vectors.
Notice that the trace of the tangential components of the functions $\phi^{e_j}_{k,q}$ coincides with the Legendre polynomials $L_0, ..., L_{p^{e_j}}$ on edge e_j.
Bubble functions:

$$
\begin{aligned}
\phi^{b,1}_{k_1,k_2,q} &= L_{k_1}(\xi_1) l_{k_2}(\xi_2)\mathbf{e}_1, \quad 0 \le k_1 \le p_1, 2 \le k_2 \le p_2 + 1 \\
\phi^{b,2}_{k_1,k_2,q} &= l_{k_1}(\xi_1) L_{k_2}(\xi_2)\mathbf{e}_2, \quad 2 \le k_1 \le p_1 + 1, 0 \le k_2 \le p_2
\end{aligned}
$$

Example: Curl-Curl Operator in $(-1,1)^2$

Curl-curl product in $(-1,1)^2$

$$
\begin{aligned}
(\phi^{b,1}_{i,j}, \phi^{b,2}_{k,l})_{H(\mathrm{curl})(K_q)} &= \int_{K_q} \mathrm{curl}\phi^{b,1}_{i,j}(x,y) \cdot \mathrm{curl}\phi^{b,2}_{k,l}(x,y)\, \mathrm{d}x\mathrm{d}y \\
&= -\int_{K_q} L_i(x) l'_j(y) l'_k(x) L_l(y) \mathrm{d}x\mathrm{d}y \neq 0
\end{aligned}
$$

L^2-product in $(-1,1)^2$

$$
\begin{aligned}
(\phi^{b,1}_{i,j}, \phi^{b,1}_{k,l})_{L^2(K_q)} &= \int_{K_q} L_i(x) l_j(y) L_k(x) l_l(y) \mathrm{d}x\mathrm{d}y \\
&= \delta_{ik} \int_{-1}^{1} l_j(y) l_l(y) \mathrm{d}y \neq \delta_{ik}\delta_{jl}
\end{aligned}
$$

Improvement: Generalized Eigenproblem for Curl-Curl Operator

Finite edge element K is associated with the polynomial space $\boldsymbol{Q}_p$.

$$\boldsymbol{Q}_p = \{\boldsymbol{E} \in Q_{p_1-1,p_2} \times Q_{p_1,p_2-1};\ \boldsymbol{E}\cdot\tau|_{e_j} \in P_p(e_j),\ j = 1,...,4\}$$

Dimension dim $\boldsymbol{Q}_p = (p_1 - 1)p_2 + p_1(p_2 - 1) = 2p_1p_2 - p_1 - p_2$
Solve the eigen-problem for Curl-Curl operator: find $\phi \in \boldsymbol{Q}_p(K)$ such that

$$\int_K \mathrm{curl}\phi(x,y)\cdot\mathrm{curl}\psi(x,y)\,\mathrm{d}x\mathrm{d}y = \lambda\int_K \phi(x,y)\cdot\psi(x,y)\,\mathrm{d}x\mathrm{d}y \quad \forall\psi \in \boldsymbol{Q}_p(K)$$

Properties - orthogonality in both products

$$\begin{aligned}\int_K \mathrm{curl}\phi_i(x,y)\cdot\mathrm{curl}\phi_j(x,y)\,\mathrm{d}x\mathrm{d}y &= 0, \quad i \neq j\\ \int_K \phi_i(x,y)\cdot\phi_j(x,y)\,\mathrm{d}x\mathrm{d}y &= 0 \quad i \neq j\end{aligned}$$

Product Element $K_q = (-1,1)^2$

On the product element we can express the eigen functions of the Curl-Curl operator with the aid of the eigenfunctions for Laplace operator in 1D.

Lemma
Let $\varphi_i^{p_1}(x), i = 2, \ldots, p_1$ and $\varphi_j^{p_2}(x), j = 2, \ldots, p_2$ be the generalized eigenfunctions of Laplace operator in 1D.
Let us define

$$\psi_{ij}^{p_1,p_2,1}(x,y) = \left((\varphi_i^{p_1})'(x)\, \varphi_j^{p_2}(y),\ \varphi_i^{p_1}(x)\, (\varphi_j^{p_2})'(y) \right),$$

$$i = 2, \ldots, p_1,\ j = 2, \ldots, p_2 \qquad (1)$$

Then for $\psi_{ij}^{p_1,p_2,1}$ holds

$$\int_\Omega \operatorname{curl} \psi_{ij}^{p_1,p_2,1} \operatorname{curl} \psi_{kl}^{p_1,p_2,1} = 0, \quad \& \quad \int_\Omega \psi_{ij}^{p_1,p_2,1} \cdot \psi_{kl}^{p_1,p_2,1} = D_{ik}\, D_{jl}.$$

We obtained

- $(p_1 - 1)(p_2 - 1) = p_1 p_2 - p_1 - p_2 + 1$ functions
- satisfy the generalized eigenproblem of curl-curl operator (correspond to zero eigenvalues)
- are defined as the gradients of the generalized eigenfunctions of Laplacian on the square $(-1,1)^2$

We have to find

- additional functions which solve generalized eigenproblem of the curl-curl operator
- to complete the basis in $\boldsymbol{Q}_p$ ($\dim \boldsymbol{Q}_p = 2p_1p_2 - p_1 - p_2$), we need $p_1p_2 - 1$ functions

Let us define

$$\psi_{ij}^{p_1,p_2,2}(x,y) = \left(-\frac{\lambda_j^{p_2}}{\lambda_i^{p_1}+\lambda_j^{p_2}}\,(\varphi_i^{p_1})'(x)\,\varphi_j^{p_2}(y),\ \frac{\lambda_i^{p_1}}{\lambda_i^{p_1}+\lambda_j^{p_2}}\,\varphi_i^{p_1}(x)\,(\varphi_j^{p_2})'(y)\right),$$

$$i = 2,\dots,p_1,\ j = 2,\dots,p_2 \qquad (2)$$

$$\psi_{i0}^{p_1,p_2,3}(x,y) = \left(0, \frac{1}{\sqrt{2}}\,\varphi_i^{p_1}(x)\right), \quad i = 2,\dots,p_1 \qquad (3)$$

$$\psi_{0j}^{p_1,p_2,3}(x,y) = \left(\frac{1}{\sqrt{2}}\,\varphi_j^{p_2}(y), 0\right), \quad j = 2,\dots,p_2 \qquad (4)$$

Lemma Functions (2), (3) and (4) satisfy

$$\int_{-1}^{1}\int_{-1}^{1} \psi_{ij}^{p_1,p_2,s}(x,y)\cdot\psi_{kl}^{p_1,p_2,1}(x,y) = 0.$$

$$\int_{-1}^{1}\int_{-1}^{1} \mathrm{curl}\ \psi_{ij}^{p_1,p_2,r}(x,y)\ \mathrm{curl}\ \psi_{kl}^{p_1,p_2,s}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \delta_{ik}\delta_{jl},$$

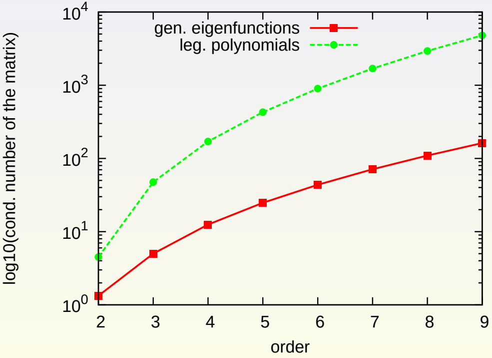
$$\int_{-1}^{1}\int_{-1}^{1} \psi_{ij}^{p_1,p_2,r}(x,y)\ \psi_{kl}^{p_1,p_2,s}(x,y)\,\mathrm{d}x\,\mathrm{d}y = D_{ik}D_{jl},$$

where $r, s = 2, 3$.

Condition Numbers - Product Element $K_q = (-1,1)^2$

As a result, we have a basis in the polynomial space Q_p, which is orthogonal in L^2 product and orthonormal in semidefinite curl-curl product and which solves the generalized eigenproblem of the curl-curl operator. It implies excellent conditioning properties. Comparison of condition numbers for the following matrix:

$$\int_{K_q} \mathrm{curl}\phi_i \, \mathrm{curl}\phi_j \, \mathrm{d}x\mathrm{d}y - \int_{K_q} \phi_i \phi_j \, \mathrm{d}x\mathrm{d}y, \qquad \phi_i, \phi_j \text{ - shape functions}$$



Example: L-Shape Domain - Diffraction problem

Refined mesh → the matrix is affected by

- the edge shape functions - both sets have same edge functions, they differ only in bubble functions
- by the shapes of the elements

But conditioning properties of the generalized eigenfunctions are still better than other used sets of shape functions.

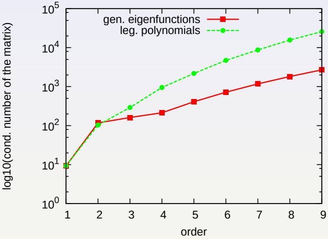
Time-harmonic Maxwell's equations with exact solution

$$\begin{aligned} \mathbf{curl}(\mathrm{curl}\boldsymbol{E}) - \boldsymbol{E} &= \boldsymbol{F} \ \text{ in } \Omega, \\ \boldsymbol{E} \cdot \tau &= 0 \ \text{ on } \Gamma_P \\ \mathrm{curl}\boldsymbol{E} - i\boldsymbol{E} \cdot \tau &= \boldsymbol{g} \cdot \tau, \ \text{ on } \Gamma_I \end{aligned}$$

Exact solution expressed by Bessel function

$$\boldsymbol{E} = \mathbf{curl}(J_\alpha(r)\cos(\alpha\phi))$$

Example: L-Shape Domain - Diffraction problem



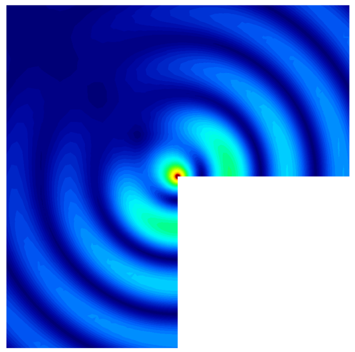


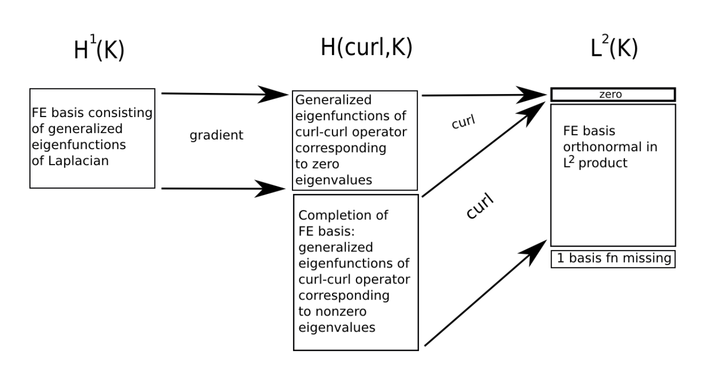
Figure: Comparison of condition numbers (left), magnitude of solution ***E***

De Rham Diagram and Finite Elements Based on Gen. Eigenfunctions

- Differential scheme relating spaces H^1, $\boldsymbol{H}(\mathrm{curl})$ and L^2:

$$H^1 \xrightarrow{\nabla} \boldsymbol{H}(\mathrm{curl}) \xrightarrow{\nabla\times} L^2$$

This class of finite elements naturally fits into the De Rham diagram.
Let $K \subset \mathbb{R}^2$. Then



Outline

Conformity Requirements of the Space $\boldsymbol{H}(\text{curl})$

Space $\boldsymbol{H}(\text{curl})$

- Best choice for discretization of Maxwell's equations (allows discontinuous functions, bigger than H^1)
- Corresponding polynomial space $\boldsymbol{H}(\text{curl}, \Omega_{hp})$ is vector-valued.
- Basis functions are associated with element edges, elements are called **edge elements**
- In contrary to the space H_1, in $\boldsymbol{H}(\text{curl})$ there are no vertex basis functions: easier to handle hanging nodes.

Conformity requirements of the space $\boldsymbol{H}(\text{curl}, \Omega_{hp})$:

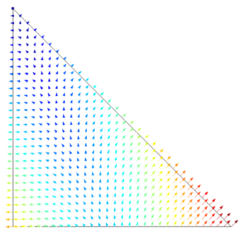
suppose $\boldsymbol{E} \in \boldsymbol{H}(\text{curl}, \Omega_{hp})$

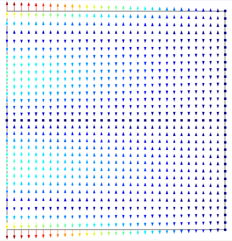
- $\boldsymbol{E}|_K \in \left[H^1(K)\right]^2$ for each element K
- For each edge $e = K_1 \cap K_2$ the traces of the tangential components are same:

$$\boldsymbol{E}|_{K_1} \times \boldsymbol{t}_e = \boldsymbol{E}|_{K_2} \times \boldsymbol{t}_e$$

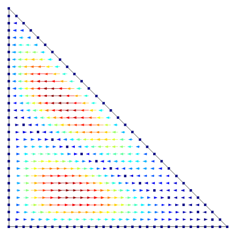
Shape Functions

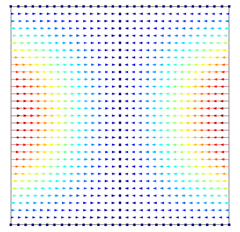
- no vertex basis functions
- $p + 1$ edge functions for polynomial degree p (associated with edge)





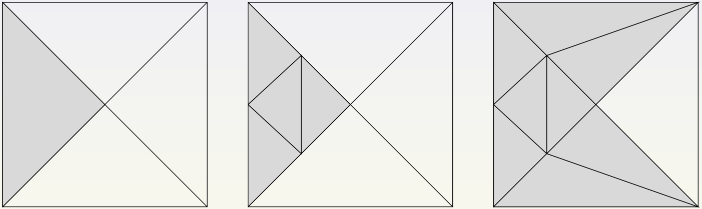
- bubble functions (starting at pol. degreee 2)





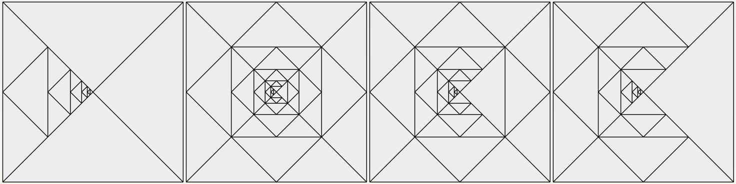
Why Hanging Nodes?

- Mesh refinement



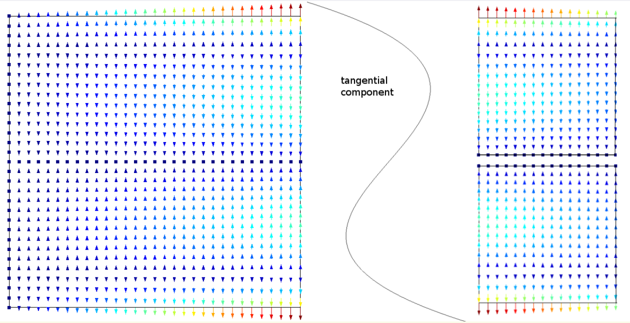
Why Hanging Nodes?

Free mesh, 1-irregular, 2-irregular, 3-irregular:



Hanging nodes

- edges:
  - egde shape functions have to satisfy conformity requirements = tangential components have to be the same at the interface of two elements
  - for edge element with pol. degree p – tangential components of all edge shape functions are of degree $0, 1, \ldots, p$
  - tangential component of each part of the constraining edge function has to be expressed by a linear combination of tangential components of edge functions, which are attached to this part of the edge $\Rightarrow$ tangential components will be the same
  - all these linear combinations of edge shape functions together with the constraining function make up one basis function, which fulfill conformity requirements of the space $\boldsymbol{H}(\mathrm{curl})$.



- bubbles:
  - no problem with bubble functions - the support is only one element, no conformity requirements

Test Example: L-Shape Domain

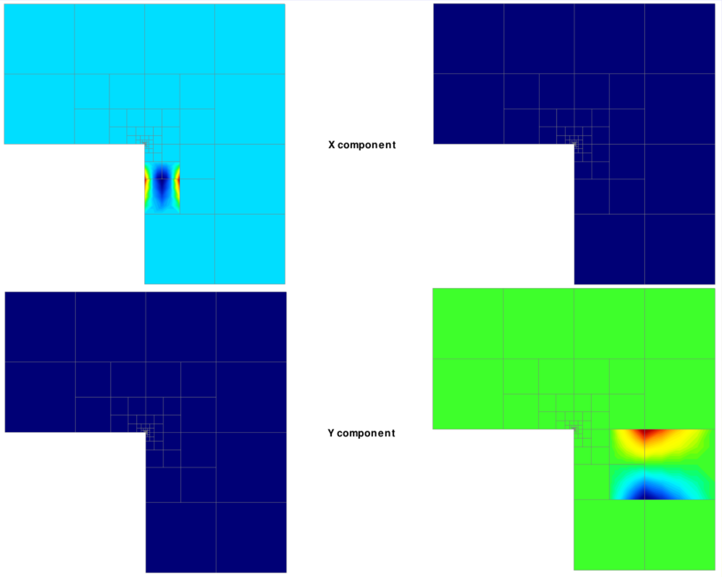
Time-harmonic Maxwell's equations with exact solution

$$\begin{aligned} \mathbf{curl}(\mathrm{curl}\boldsymbol{E}) - \boldsymbol{E} &= \boldsymbol{F} \ \text{ in } \Omega, \\ \boldsymbol{E} \cdot \tau &= 0 \ \text{ on } \Gamma_P \\ \mathrm{curl}\boldsymbol{E} - i\boldsymbol{E} \cdot \tau &= \boldsymbol{g} \cdot \tau, \ \text{ on } \ \Gamma_I \end{aligned}$$

Exact solution

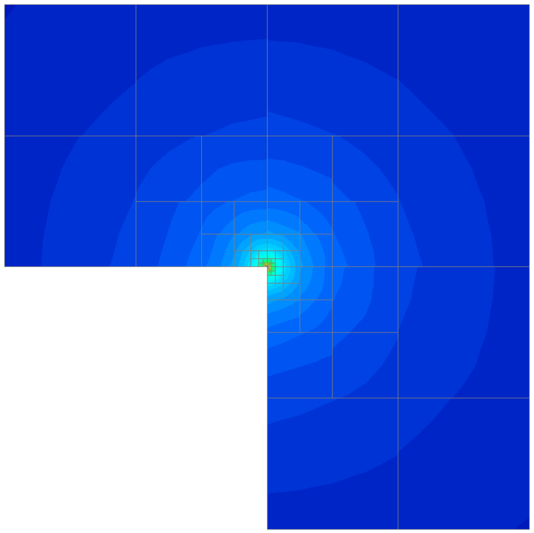
$$\boldsymbol{E} = 2/3 r^{-1/3} \begin{bmatrix} \cos(\frac{\pi}{6} + \frac{\theta}{3}) \\ \sin(\frac{\pi}{6} + \frac{\theta}{3}) \end{bmatrix}$$

Test Example: Basis functions



Test Example: L-Shape Domain

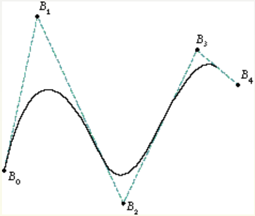
Magnitude of solution ***E***

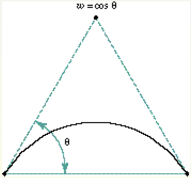


Outline

NURBS Curves

- can accurately represent standard geometric objects like lines, circles, ellipses, parabolas and hyperbolas,
- are generalizations of non-rational B-splines and non-rational and rational Bezier curves,
- can be evaluated reasonably fast by numerically stable and accurate algorithms,
- defined by three things:
  - degree k (usually 1,2,3,5),
  - control points P_i, weights w_i, $\quad i = 0, \dots n-1$,
  - knot vector $\{t_0, \dots, t_{m-1}\}$.





Definition of NURBS Curve

NURBS curve is a curve parametrized by $C(t)$, $t \in [0, 1]$

$$C(t) = \frac{\sum_{i=0}^{n-1} w_i \, P_i \, N_{i,k}(t)}{\sum_{i=0}^{n-1} w_i \, N_{i,k}(t)} \tag{5}$$

where $N_{i,k}$ is the B-spline basis function of degree k defined recursively

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t) \tag{6}$$

$$N_{i,0}(t) = \begin{cases} 1 & \text{if} t_i \leq t < t_{i+1} \\ 0 & \text{else} \end{cases} \tag{7}$$

Knot vector $\{t_0, \ldots, t_{m-1}\}$, $\quad m = k + n + 1$ (degree + # of control points + 1):

- nondecreasing sequence $t_i \leq t_{i+1}$, spacing can be nonuniform
- first $k + 1$ components is 0.0 and last $k + 1$ is equal to 1.0 (for example $(0, 0, 0, 0.2, 0.8, 1, 1, 1)$ for degree 2 and 5 control points)

Example: Part of a Circle

Construction of a circular arc with angular extent α above edge AB.

- degree: 2
- # of control point: 3 ($P_0 = A$, $P_1 =$ see in figure, $P_2 = B$)
- weights: $w_0 = w_2 = 1.0$, inner $w_1 = \cos\phi$
- knot: $(0, 0, 0, 1, 1, 1)$

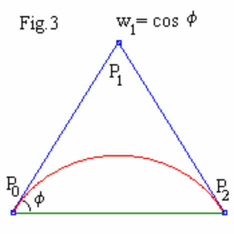
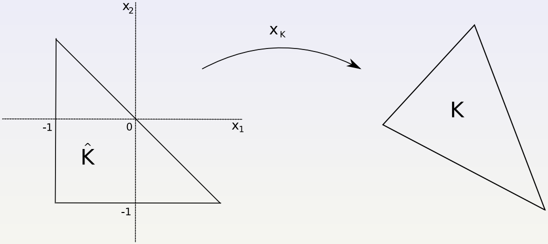


Figure: Construction of circular arc, $\phi = \alpha/2$

Reference Mapping

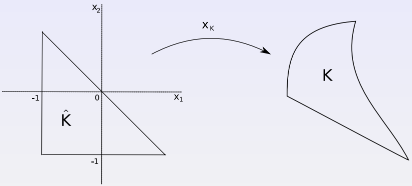
Reference mapping from reference domain into physical element



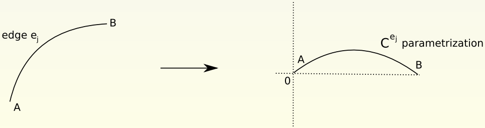
- for elements with straight edges is linear
- for curvilinear elements (defined with NURBS) is even nonpolynomial - edges are parametrized by nonpolynomial functions (NURBS).

Reference Mapping

Reference mapping for curvilinear elements: $X_K = X_K^v + X_K^e$



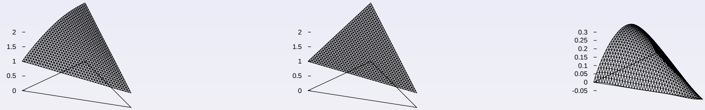
- affine part is the same as for straight elements: $X_K^v = \sum_{i=1}^{3} x_i \varphi^{v_j}$ (physical mesh vertices and vertex functions)
- higher order part: $X_K^e = \sum_{j=1}^{3} C^{e_j} \lambda_A \lambda_B$, where C^{e_j} is the parametrization of the curved edge (straight part was subtracted) and λ_A, λ_B are vertex functions corresponding to vertices A and B.



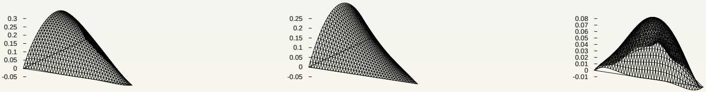
Isoparametric Approximation

isoparametric approximation = polynomial map defined in terms of master element shape functions

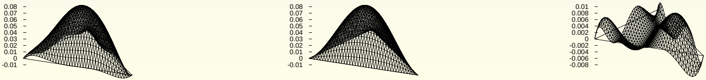
- Vertex interpolant $X^v_{h,p}$ - linear combination of the vertex shape functions



- Edge interpolant $X^e_{h,p}$ - by projecting the residual $X - X^v_{h,p}$ on the polynomial space generated by the edge functions



- Bubble interpolant $X^b_{h,p}$ - by projecting the residual $X - X^v_{h,p} - X^e_{h,p}$ on the polynomial space generated by the bubble functions



Numerical Quadrature and Curvilinear Elements

Example: transformation to the reference domain

$$\int_K \nabla\varphi_i \, \nabla\varphi_j \, \mathrm{d}x = \int_{\hat{K}} J_{x_K} \, (\frac{\mathrm{d}x_K}{\mathrm{d}\xi})^{-1} \, \nabla\hat{\varphi}_i \, (\frac{\mathrm{d}x_K}{\mathrm{d}\xi})^{-1} \, \nabla\hat{\varphi}_j \, \mathrm{d}\xi$$

To integrate this expresion

- determine polynomial degree $=$ polynomial degree of $\hat{\varphi}_i +$ polynomial degree of $\hat{\varphi}_j +$ additional degree necessary for the inverse mapping and Jacobian
- use accurate quadrature rule

- for linear triangles reference mapping is constant – OK (no additional degree)
- for quads and arbitrary curvilinear elements the inverse reference mapping is even nonpolynomial (for example: quads with small angles, curved elements with angle almost $180^\circ, \ldots$)
  - try better and better quadrature rule until the integral converges
  - if we don't have enough quadrature rules $\Rightarrow$ warn the user
  - or use adaptive quadrature in critical elements - refine the element domain until the integration error is small

Triangular/Quadrilateral Meshes

Triangles:

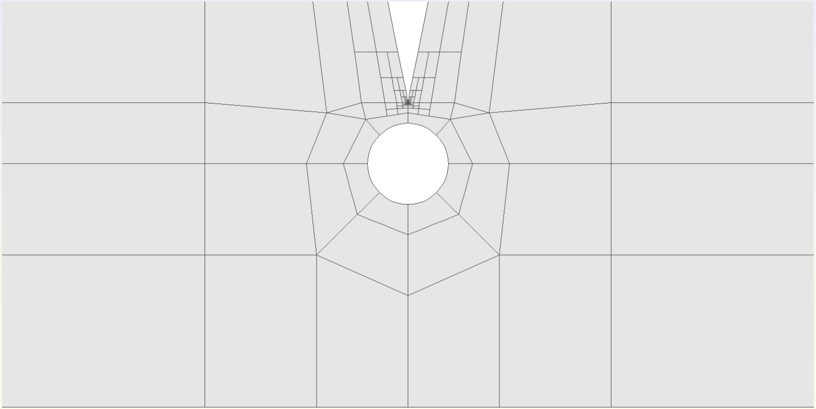
- good for approximation of solution that has the same behaviour in all directions
- reference mapping from master domain onto physical elements is constant ⇒ no additional quadrature rules
- quadrature rules are known only up to degree 20 (limits for polynomial degree on elements)

Quads

- allow different polynomial degrees in different directions (good for approximation of solution in boundary layers)
- inverse reference mapping is not even polynomial - problems with sufficent quadrature rule (integrals converge very slowly)
- we can use product of quadrature rules in 1D (no limits for polynomial degree)

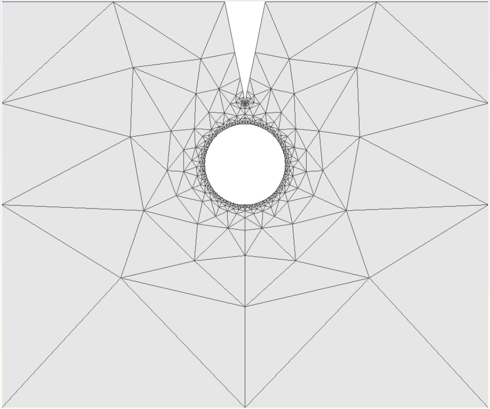
Sphere-Cone Problem - H^1 problem

A metallic sphere carries an electric potencial $\varphi_S = 100kV$. To describe the surface of the sphere we need only 8 curvilinear elements. The mesh is refined near the peak of the cone (singularity of the gradient of the solution)



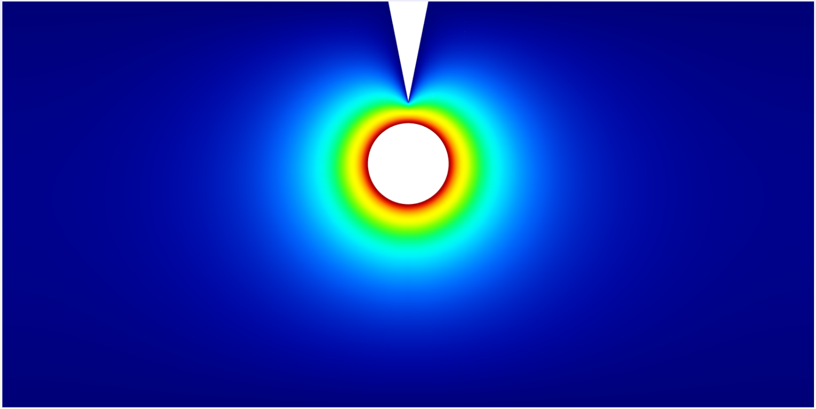
Sphere-Cone Problem

To describe the surface of the sphere we need hundreds of straight elements (only because of the curved boundary - not necessary for the solution).



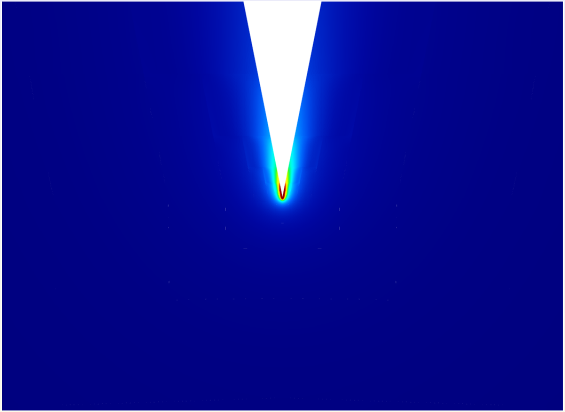
Sphere-Cone Problem

Solution



Sphere-Cone Problem

Gradient of the solution



Thank you for your attention